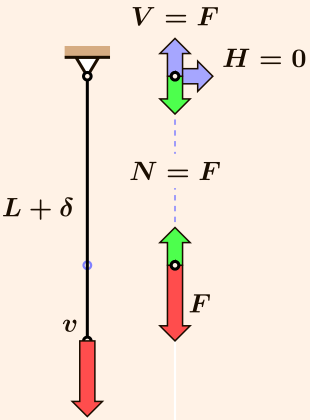
Sólido deformable: compatibilidad y equilibrio (3rd ed)

Mariano Vázquez Espí

Madrid (España), 15 de marzo de 2017.

Modelo ‘cable’



Modelo ‘cable’

Equilibrio

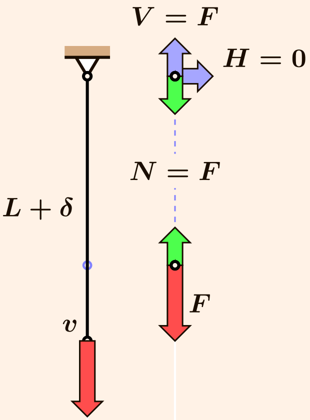
global

$$F = V \qquad H = 0$$

interno

$$N = \sigma A$$

$$N = F$$



Modelo ‘cable’

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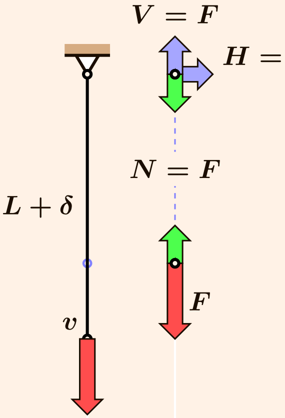


Compatibilidad

$$\delta = \varepsilon L$$

$$\delta = v$$

Modelo ‘cable’

Equilibrio

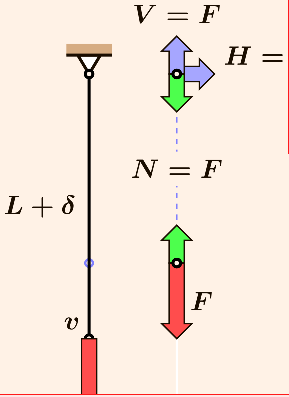
global

$$F = V \qquad H = 0$$

interno

$$N = \sigma A$$

$$N = F$$



Compatibilidad

$$\delta = \varepsilon L$$

$$\delta = v$$

Material

$$\sigma = \begin{cases} \varepsilon \mathbf{E} \text{ si } 0 \leq \varepsilon \leq \varepsilon_e \\ \mathbf{f_u} \text{ si } \varepsilon_e \leq \varepsilon \leq \varepsilon_\mathbf{u} \\ 0 \text{ en cualquier otro caso} \end{cases}$$

Modelo ‘cable’

Equilibrio

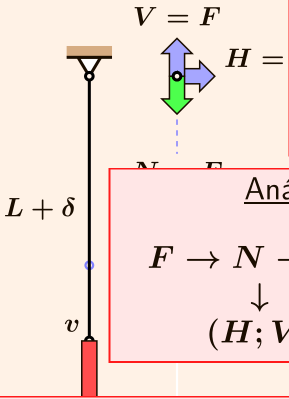
global

$$F = V \qquad H = 0$$

interno

$$N = \sigma A$$

$$N = F$$



Compatibilidad

$$\delta = \varepsilon L$$

$$\delta = v$$

Análisis isostático

$$F \to N \to \sigma \to \varepsilon \to \delta \to v$$
$$\downarrow$$
$$(H; V)$$

Material

$$\sigma = \begin{cases} \varepsilon \mathbf{E} \text{ si } 0 \leq \varepsilon \leq \varepsilon_e \\ \mathbf{f_u} \text{ si } \varepsilon_e \leq \varepsilon \leq \varepsilon_\mathbf{u} \\ 0 \text{ en cualquier otro caso} \end{cases}$$

Ecuaciones de equilibrio

Son las relaciones entre las fuerzas interiores de los sólidos deformables y las acciones sobre la estructura bajo la condición de equilibrio estático.

En su forma más básica expresan las resultantes $\boldsymbol{H}, \boldsymbol{V}, \boldsymbol{M}, \ldots$ de los esfuerzos $\boldsymbol{N_1}, \boldsymbol{N_2}, \boldsymbol{N_3}, \ldots$. Cada condición de equilibrio $(\boldsymbol{H}, \boldsymbol{V}, \boldsymbol{M}, \ldots)$ resulta en una ecuación.

En estructuras sometidas a requisitos de rigidez con distorsiones tolerables muy pequeñas, se pueden escribir a partir de la geometría indeformada de la estructura sin perdida significativa de precisión, resultando en ecuaciones lineales.

Ecuaciones de compatibilidad

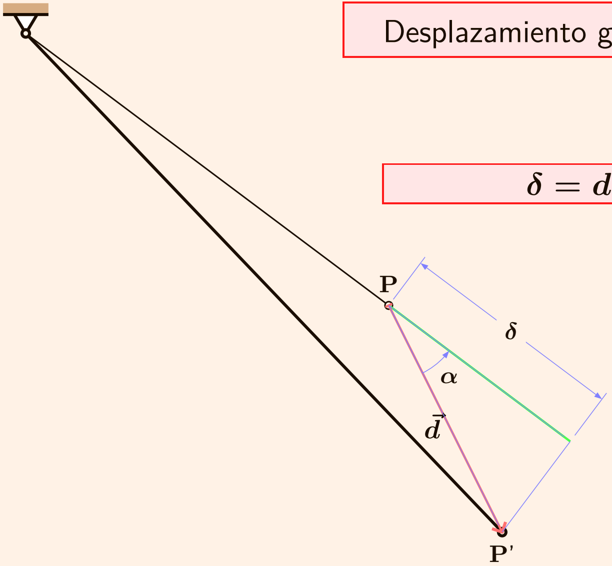
Son las relaciones entre movimientos de puntos o sólidos indeformables y la deformación de los sólidos deformables. Expresan la coherencia topológica y geométrica entre las distintas partes de una estructura y son validas siempre que la estructura no se rompa, sin importar el estado elástico o plástico de sus partes.

En su forma más básica relacionan movimientos $\boldsymbol{u, v, \theta, \dots}$ con alargamientos $\boldsymbol{\delta_1, \delta_2, \dots}$

En estructuras sometidas a requisitos de rigidez con distorsiones tolerables muy pequeñas, se pueden escribir como relaciones lineales de forma casi exacta (hipótesis de pequeñas deformaciones).

Se deducen de dibujos para cada $\boldsymbol{u, v, \theta, \dots}$

Ecuaciones de compatibilidad



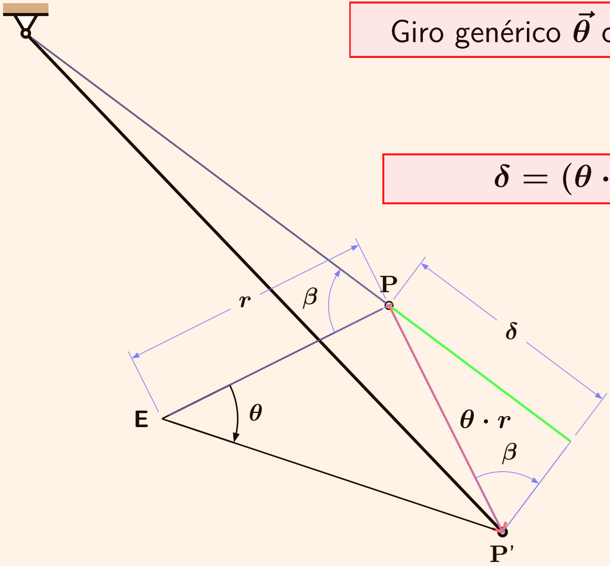
Desplazamiento genérico $\vec{d}$ de $\mathbf{P}$

$$\delta = d\cos\alpha$$

Ecuaciones de compatibilidad

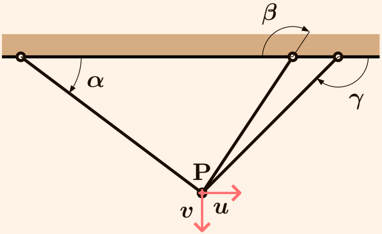
Giro genérico $\vec{\theta}$ con centro en **E**

$$\boldsymbol{\delta = (\theta \cdot r)\sin\beta}$$



Ecuaciones de compatibilidad

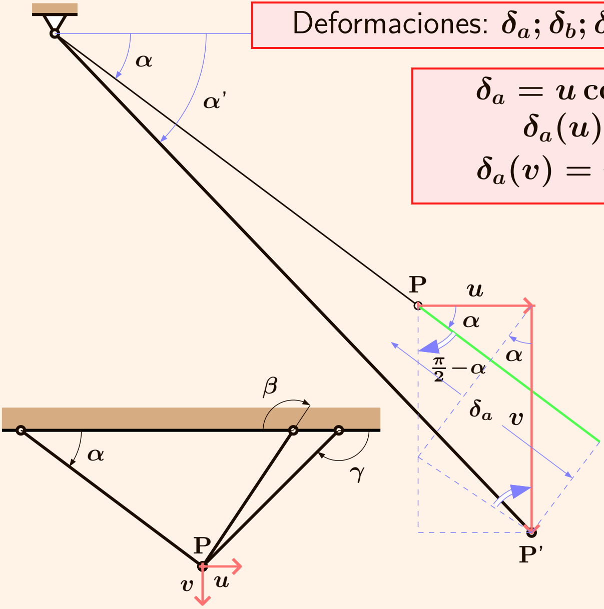
Deformaciones: $\delta_a; \delta_b; \delta_c$. Movimientos: $u; v$



Ecuaciones de compatibilidad

Deformaciones: $\delta_a; \delta_b; \delta_c$. Movimientos: $u; v$

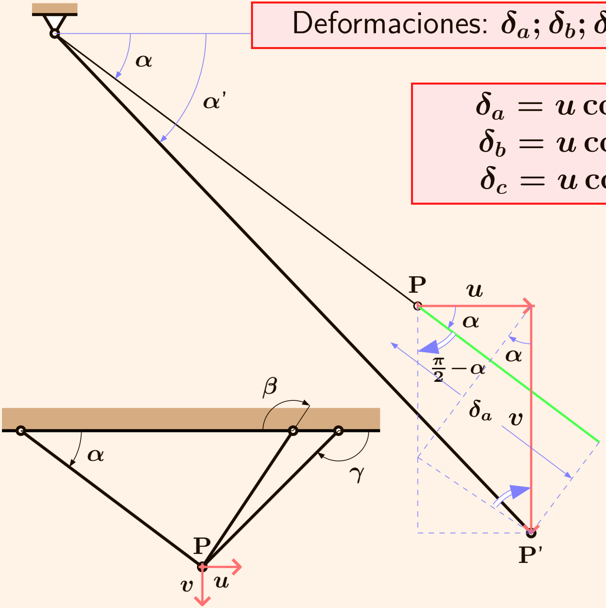
$$\delta_a = u\cos\alpha + v\sin\alpha$$
$$\delta_a(u) = u\cos\alpha$$
$$\delta_a(v) = v\cos\left(\frac{\pi}{2} - \alpha\right)$$



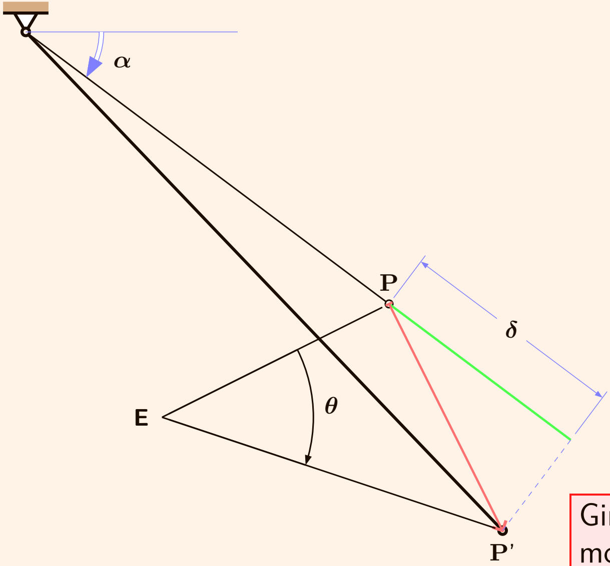
Ecuaciones de compatibilidad

Deformaciones: $\delta_a; \delta_b; \delta_c$. Movimientos: $u; v$

$$\delta_a = u\cos\alpha + v\sin\alpha$$
$$\delta_b = u\cos\beta + v\sin\beta$$
$$\delta_c = u\cos\gamma + v\sin\gamma$$

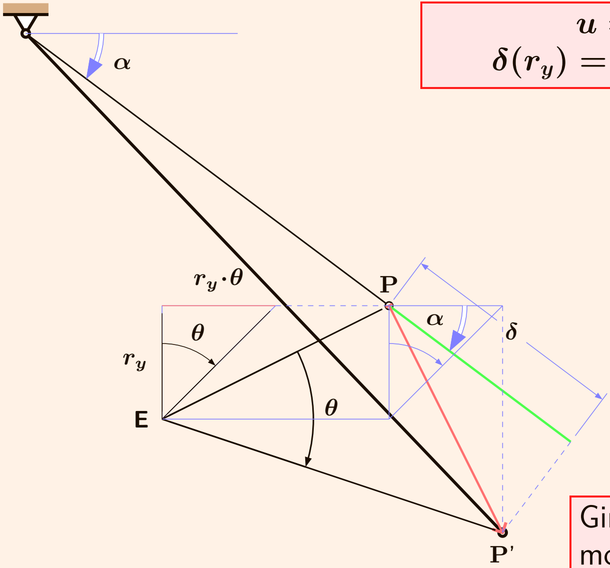


Ecuaciones de compatibilidad



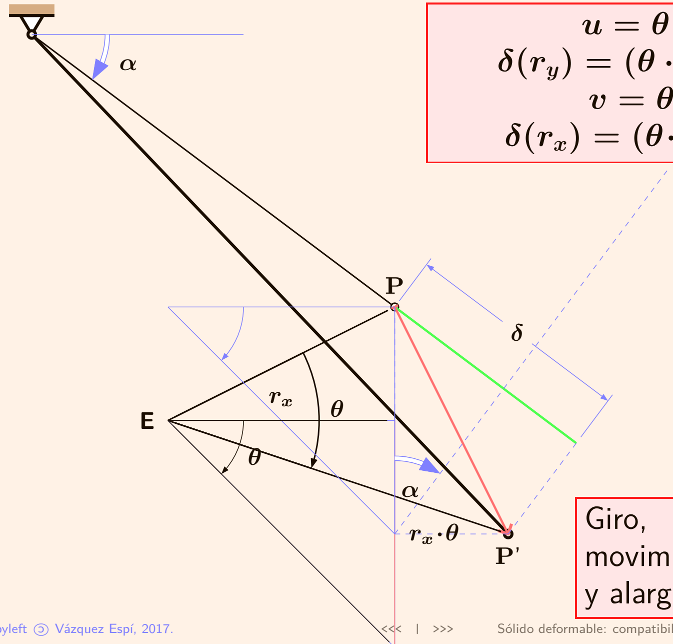
Giro,
movimientos u, v
y alargamientos

Ecuaciones de compatibilidad



$$u = \theta \cdot r_y$$
$$\delta(r_y) = (\theta \cdot r_y) \cos \alpha$$

Giro, movimientos u, v y alargamientos

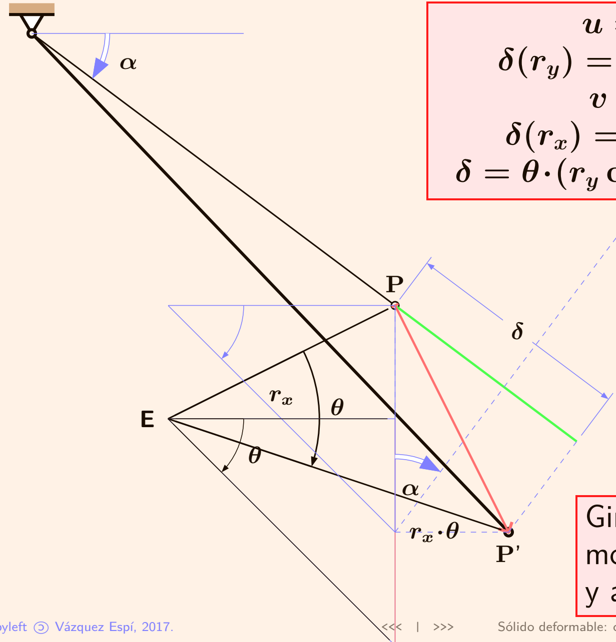


$$u = \theta \cdot r_y$$
$$\delta(r_y) = (\theta \cdot r_y)\cos\alpha$$
$$v = \theta \cdot r_x$$
$$\delta(r_x) = (\theta \cdot r_x)\sin\alpha$$

Giro,
movimientos u, v
y alargamientos

Ecuaciones de compatibilidad

$$u = \theta \cdot r_y$$
$$\delta(r_y) = (\theta \cdot r_y) \cos \alpha$$
$$v = \theta \cdot r_x$$
$$\delta(r_x) = (\theta \cdot r_x) \sin \alpha$$
$$\delta = \theta \cdot (r_y \cos \alpha + r_x \sin \alpha)$$



Giro,
movimientos u, v
y alargamientos

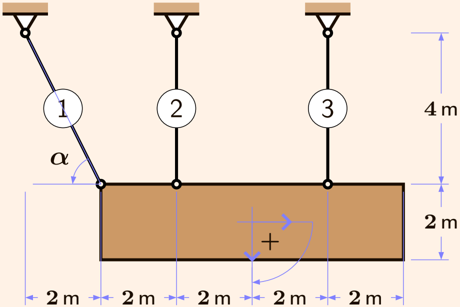
Algunos ejemplos

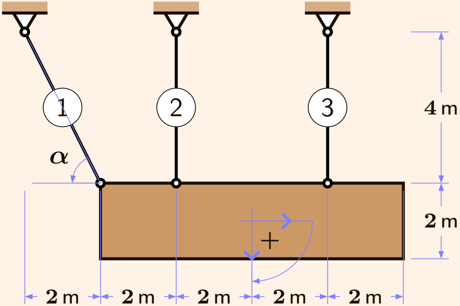
En los ejemplos siguientes tres cables sostienen una piedra con distintas geometrías y sustentación.

La piedra en principio tiene tres grados de libertad, $\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{\theta}$. Para cada uno de ellos se realiza un dibujo, suponiendo valores nulos para los otros dos. Y en cada dibujo, o bien se deduce que el grado de libertad es incompatible con la sustentación, o bien se deducen su relación con los alargamientos de los cables.

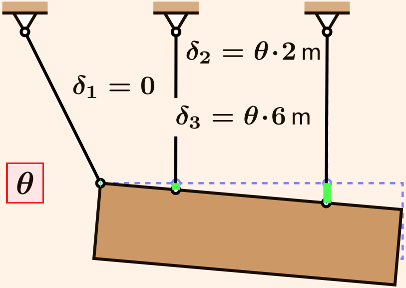
Las ecuaciones de compatibilidad, finalmente, se forman sumando las contribuciones de cada grado a cada alargamiento.

Las ecuaciones de equilibrio son las "traspuestas" de las anteriores, aunque se pueden deducir directamente.

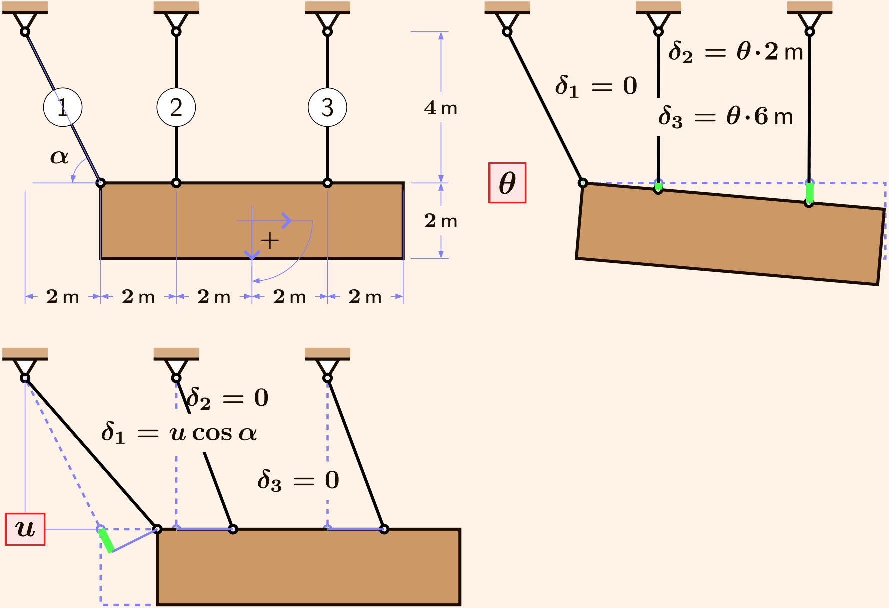

1
2
3
α
4 m
2 m
2 m
2 m
2 m
2 m
2 m
+

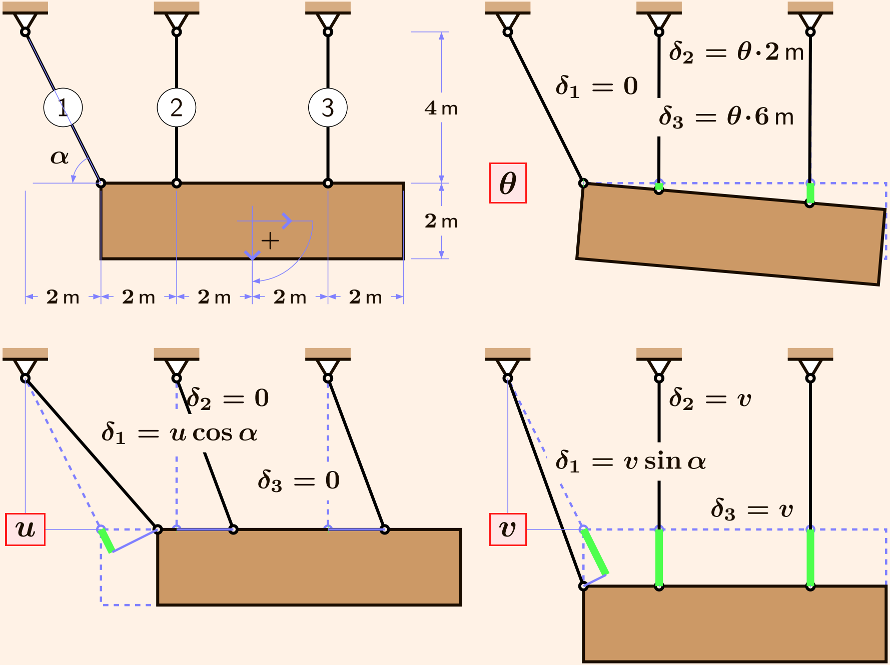


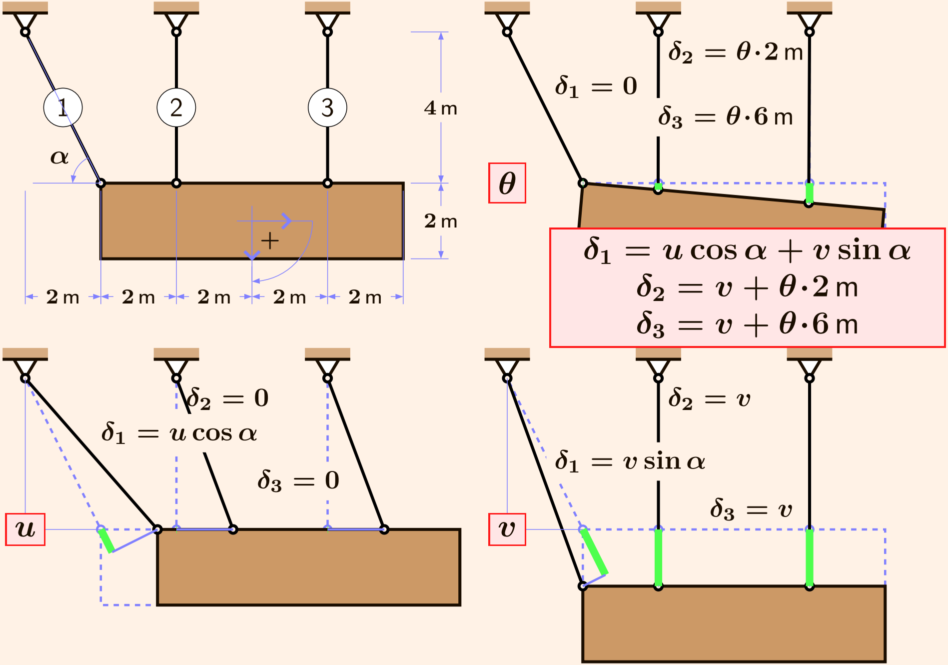
1
2
3
4 m
2 m
α
+
2 m
2 m
2 m
2 m
2 m



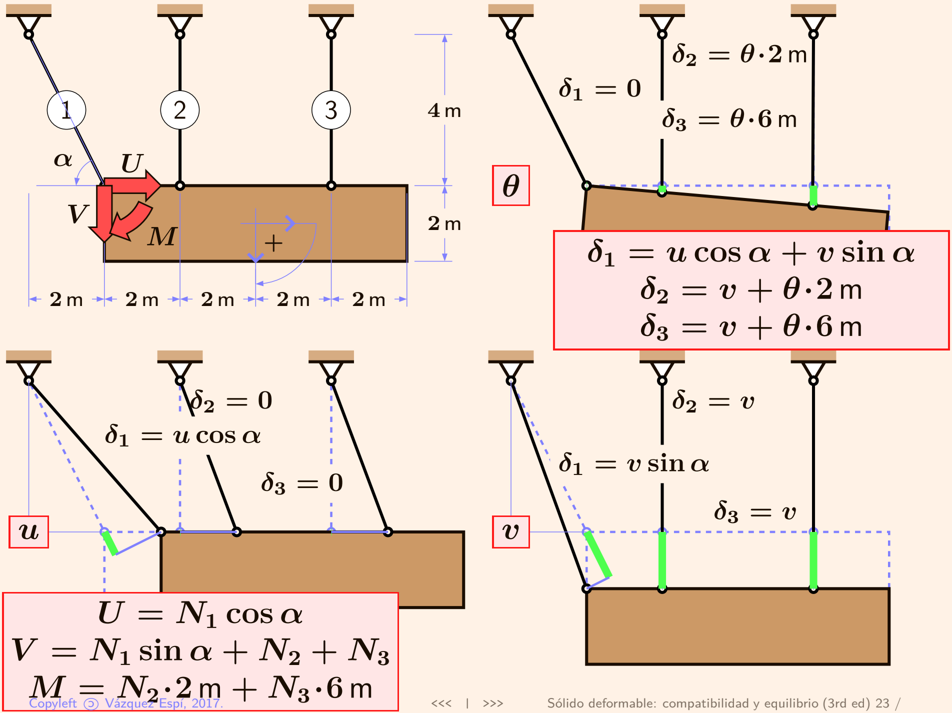
$\delta_2 = \theta \cdot 2$ m
$\delta_1 = 0$
$\delta_3 = \theta \cdot 6$ m
θ


1
2
3
α
4 m
2 m
2 m
2 m
2 m
2 m
2 m
$\delta_1 = 0$
$\delta_2 = \theta \cdot 2$ m
$\delta_3 = \theta \cdot 6$ m
θ
$\delta_2 = 0$
$\delta_1 = u \cos \alpha$
$\delta_3 = 0$
u


1
2
3
4 m
2 m
α
2 m
2 m
2 m
2 m
2 m
+
θ
δ₁ = 0
δ₂ = θ·2 m
δ₃ = θ·6 m
u
δ₁ = u cos α
δ₂ = 0
δ₃ = 0
v
δ₁ = v sin α
δ₂ = v
δ₃ = v

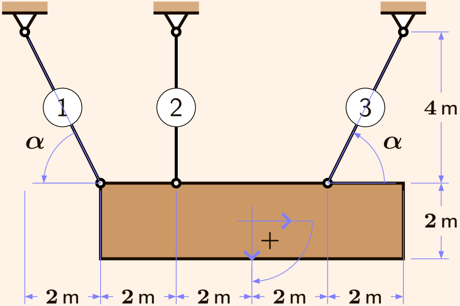


$$\delta_1 = u\cos\alpha + v\sin\alpha$$
$$\delta_2 = v + \theta \cdot 2\,\mathrm{m}$$
$$\delta_3 = v + \theta \cdot 6\,\mathrm{m}$$

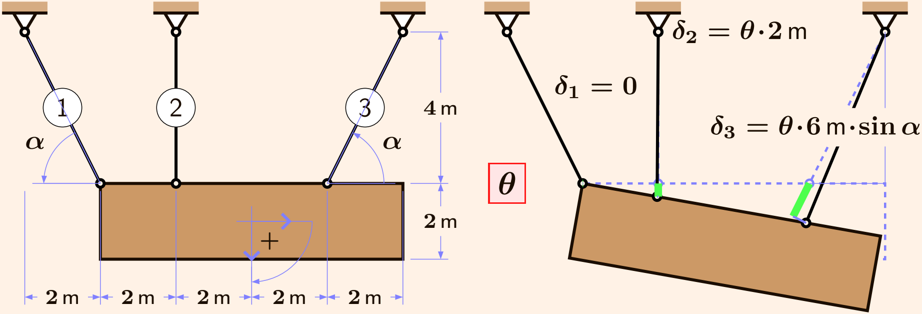


$$\delta_1 = u\cos\alpha + v\sin\alpha$$
$$\delta_2 = v + \theta\cdot 2\,\mathrm{m}$$
$$\delta_3 = v + \theta\cdot 6\,\mathrm{m}$$

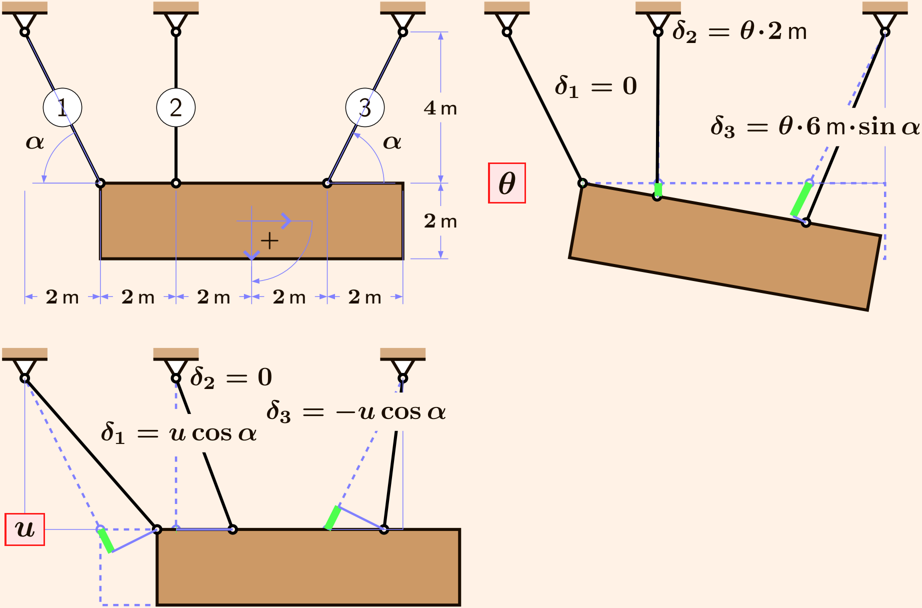
$$U = N_1\cos\alpha$$
$$V = N_1\sin\alpha + N_2 + N_3$$
$$M = N_2\cdot 2\,\mathrm{m} + N_3\cdot 6\,\mathrm{m}$$

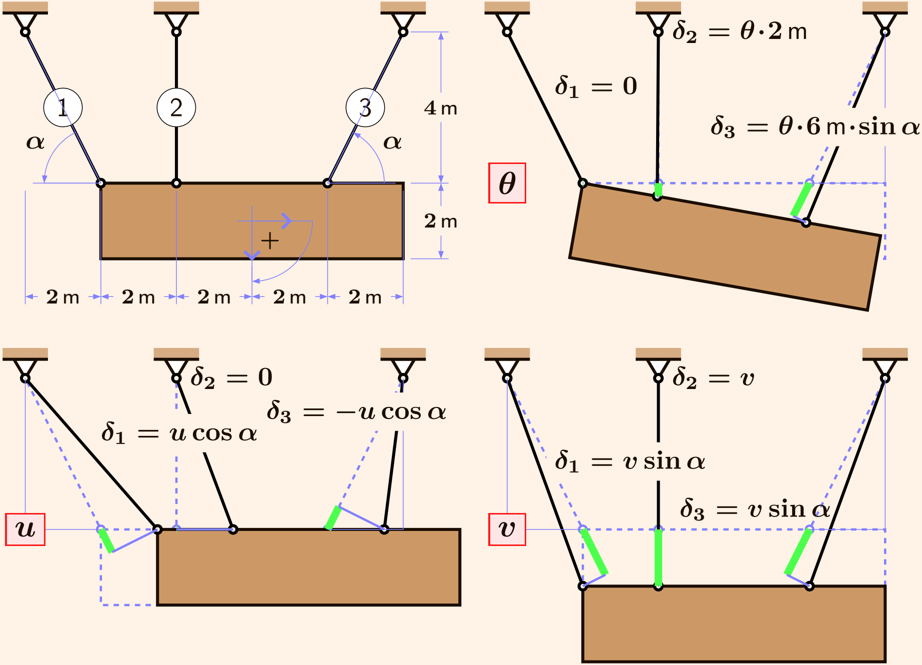


1
2
3
α
α
4 m
2 m
2 m
2 m
2 m
2 m
2 m
+

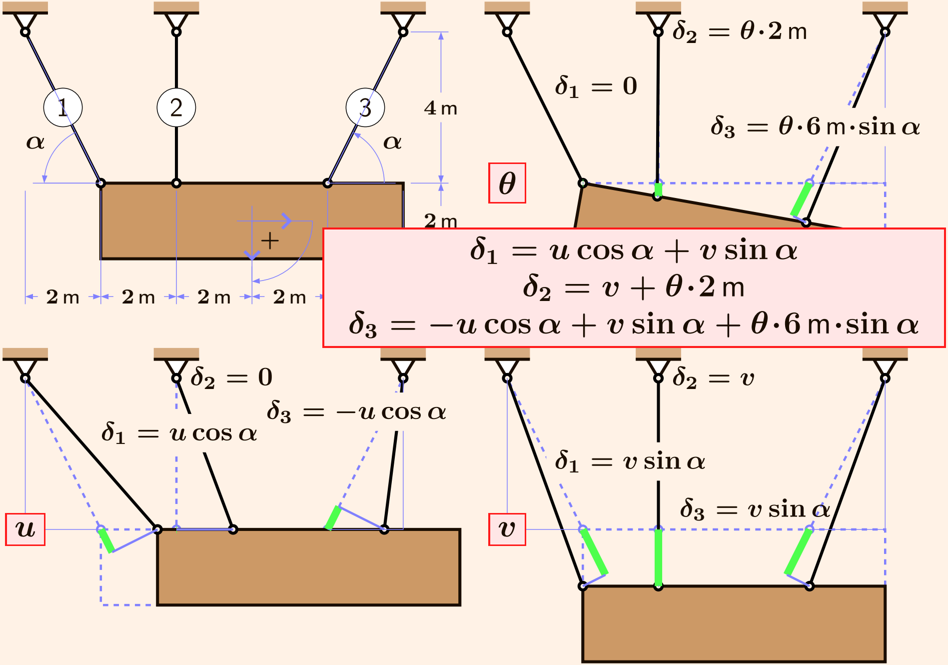


1
2
3
α
α
4 m
2 m
2 m
2 m
2 m
2 m
2 m
θ
$\delta_1 = 0$
$\delta_2 = \theta \cdot 2\,\text{m}$
$\delta_3 = \theta \cdot 6\,\text{m} \cdot \sin\alpha$

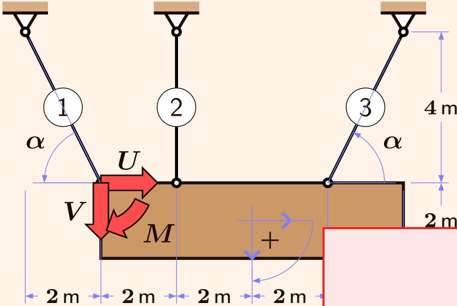

1
2
3
α
α
4 m
2 m
+
2 m
2 m
2 m
2 m
2 m
$\delta_2 = \theta \cdot 2\,\mathrm{m}$
$\delta_1 = 0$
$\delta_3 = \theta \cdot 6\,\mathrm{m} \cdot \sin\alpha$
θ
$\delta_2 = 0$
$\delta_3 = -u\cos\alpha$
$\delta_1 = u\cos\alpha$
u

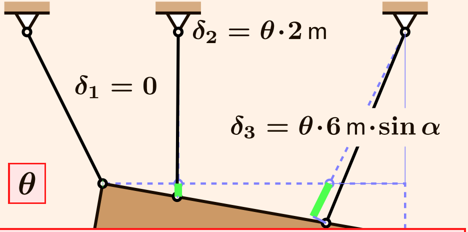


1
2
3
α
α
4 m
2 m
2 m
2 m
2 m
2 m
2 m
+
θ
δ2 = θ·2 m
δ1 = 0
δ3 = θ·6 m·sin α
u
δ2 = 0
δ1 = u cos α
δ3 = −u cos α
v
δ2 = v
δ1 = v sin α
δ3 = v sin α

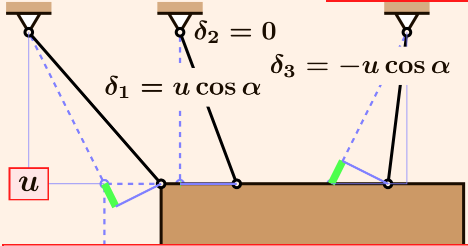


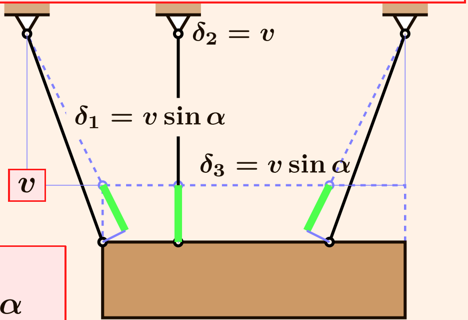
$$\delta_1 = u\cos\alpha + v\sin\alpha$$
$$\delta_2 = v + \theta\cdot 2\,\text{m}$$
$$\delta_3 = -u\cos\alpha + v\sin\alpha + \theta\cdot 6\,\text{m}\cdot\sin\alpha$$



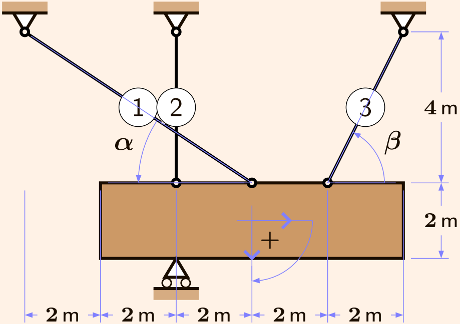


$$\delta_1 = u\cos\alpha + v\sin\alpha$$
$$\delta_2 = v + \theta \cdot 2\,\mathrm{m}$$
$$\delta_3 = -u\cos\alpha + v\sin\alpha + \theta \cdot 6\,\mathrm{m} \cdot \sin\alpha$$

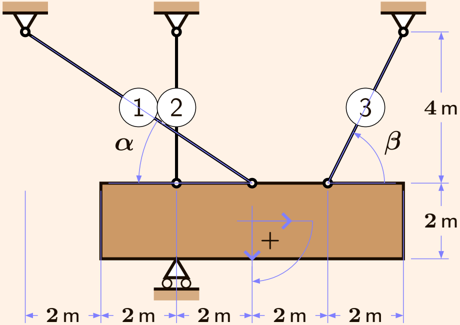


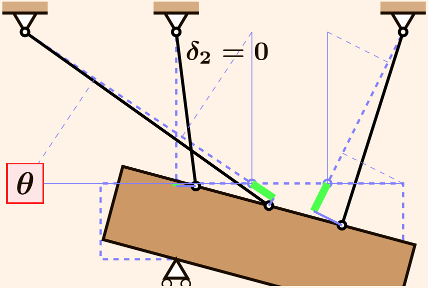


$$U = N_1\cos\alpha - N_3\cos\alpha$$
$$V = N_1\sin\alpha + N_2 + N_3\sin\alpha$$
$$M = N_2 \cdot 2\,\mathrm{m} + N_3 \cdot 6\,\mathrm{m}\sin\alpha$$

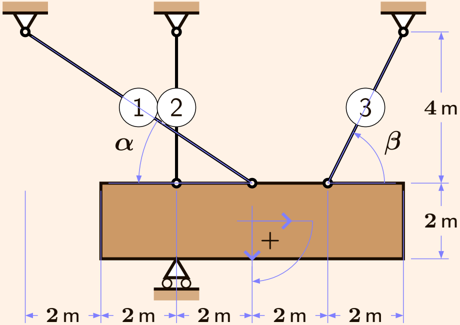


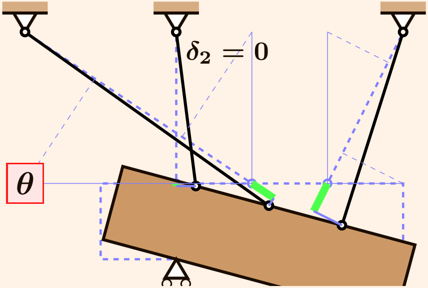
1
2
3
α
β
4 m
2 m
2 m
2 m
2 m
2 m
2 m
+



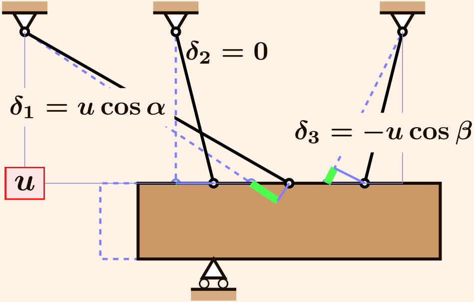


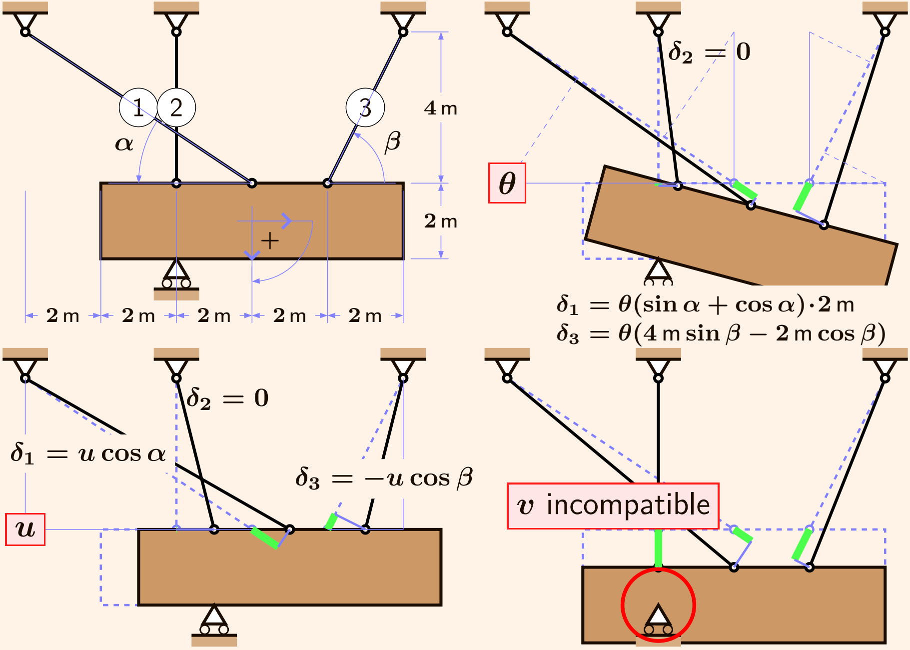
$$\delta_1 = \theta(\sin\alpha + \cos\alpha)\cdot 2\,\mathrm{m}$$
$$\delta_3 = \theta(4\,\mathrm{m}\sin\beta - 2\,\mathrm{m}\cos\beta)$$

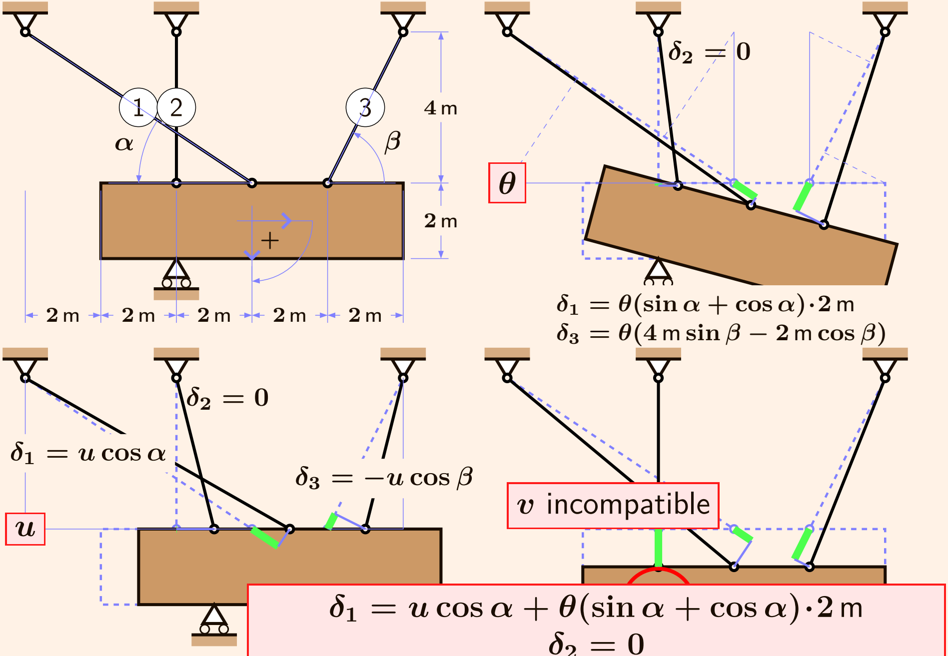




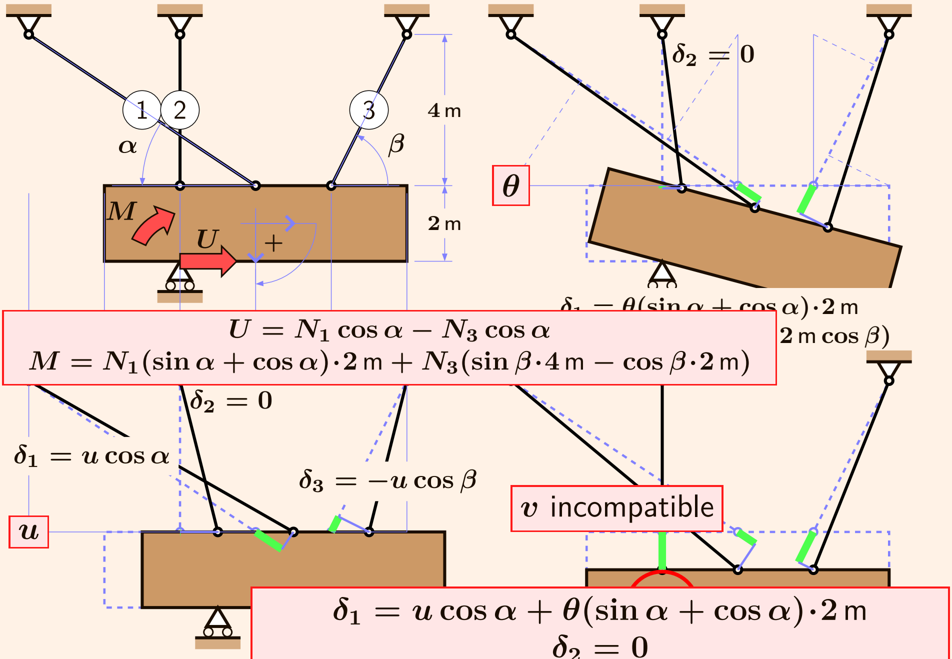
$$\delta_1 = \theta(\sin\alpha + \cos\alpha)\cdot 2\,\mathrm{m}$$
$$\delta_3 = \theta(4\,\mathrm{m}\sin\beta - 2\,\mathrm{m}\cos\beta)$$




1
2
3
α
β
4 m
2 m
+
2 m
2 m
2 m
2 m
2 m
δ₂ = 0
θ
δ₁ = θ(sin α + cos α)·2 m
δ₃ = θ(4 m sin β − 2 m cos β)
δ₂ = 0
δ₁ = u cos α
δ₃ = −u cos β
u
v incompatible

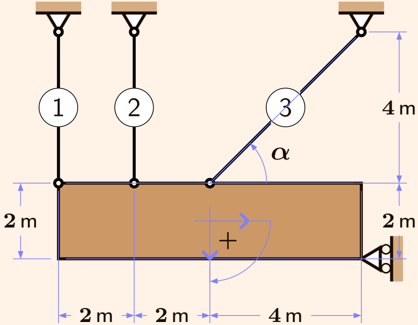


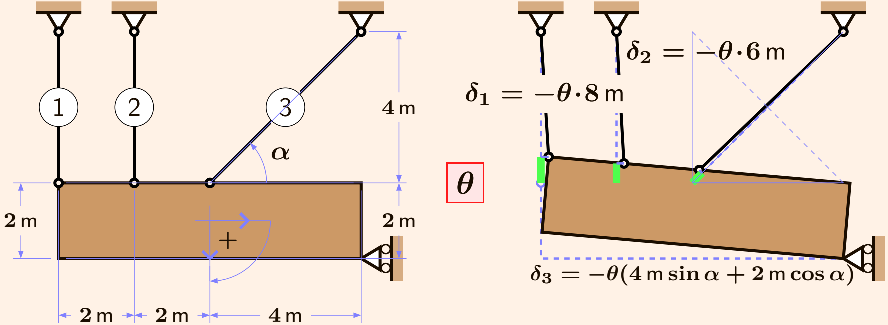
$$\delta_1 = u\cos\alpha + \theta(\sin\alpha + \cos\alpha)\cdot 2\,\mathrm{m}$$
$$\delta_2 = 0$$
$$\delta_3 = -u\cos\alpha + \theta(\sin\beta\cdot 4\,\mathrm{m} - \cos\beta\cdot 2\,\mathrm{m})$$

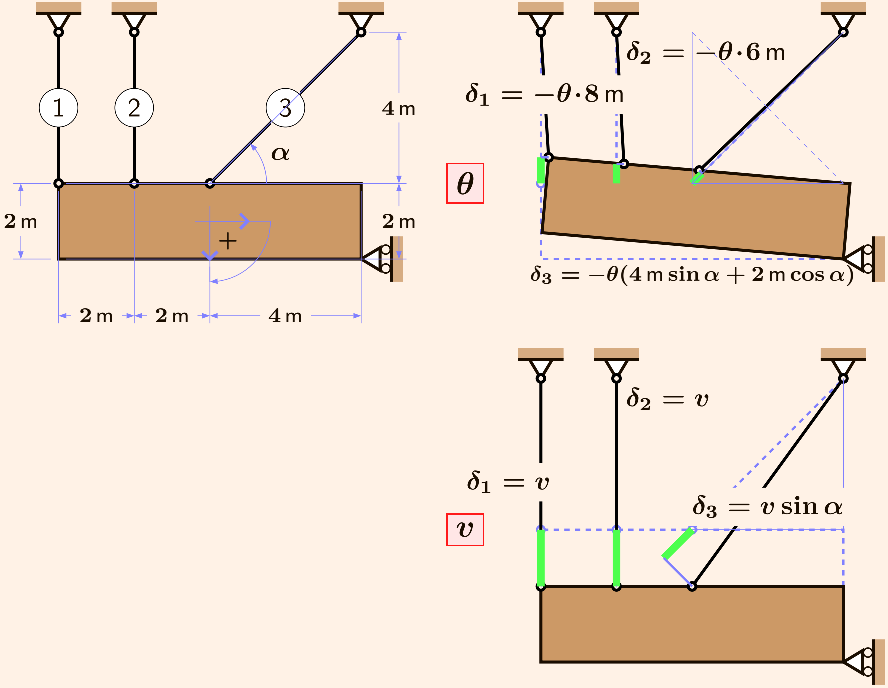


$$U = N_1 \cos\alpha - N_3 \cos\alpha$$
$$M = N_1(\sin\alpha + \cos\alpha)\cdot 2\,\mathrm{m} + N_3(\sin\beta\cdot 4\,\mathrm{m} - \cos\beta\cdot 2\,\mathrm{m})$$

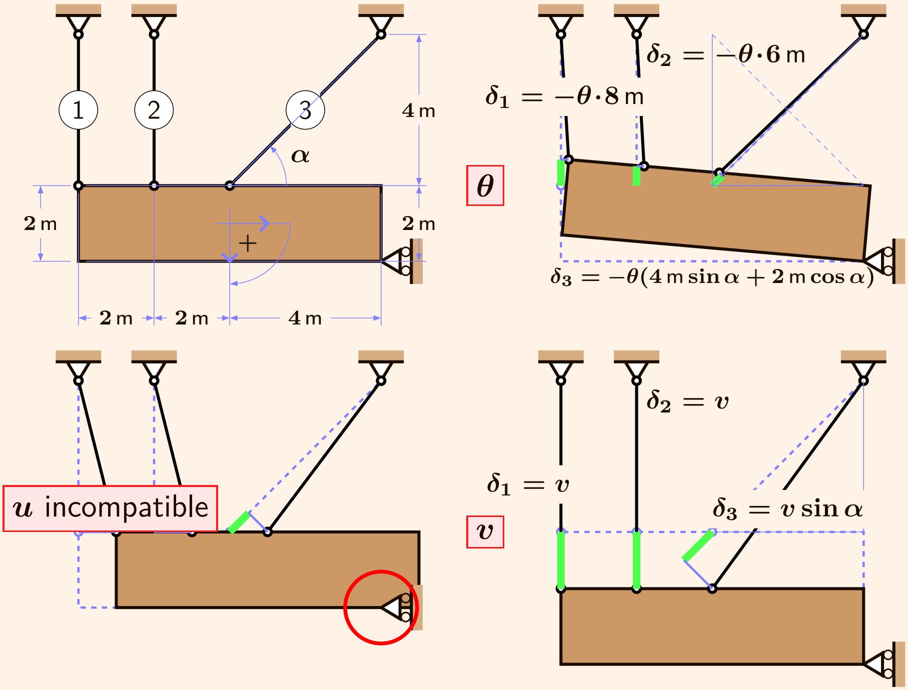
$$\delta_1 = u\cos\alpha + \theta(\sin\alpha + \cos\alpha)\cdot 2\,\mathrm{m}$$
$$\delta_2 = 0$$
$$\delta_3 = -u\cos\alpha + \theta(\sin\beta\cdot 4\,\mathrm{m} - \cos\beta\cdot 2\,\mathrm{m})$$


1
2
3
4 m
α
2 m
2 m
2 m
2 m
4 m

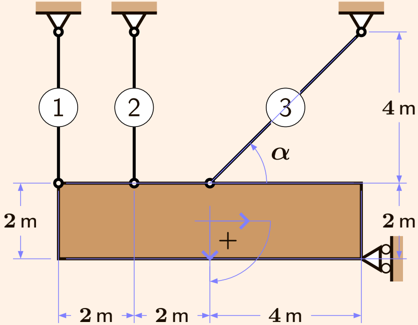

1
2
3
4 m
α
2 m
2 m
+
2 m
2 m
4 m
$\delta_2 = -\theta \cdot 6\,\mathrm{m}$
$\delta_1 = -\theta \cdot 8\,\mathrm{m}$
θ
$\delta_3 = -\theta(4\,\mathrm{m}\sin\alpha + 2\,\mathrm{m}\cos\alpha)$

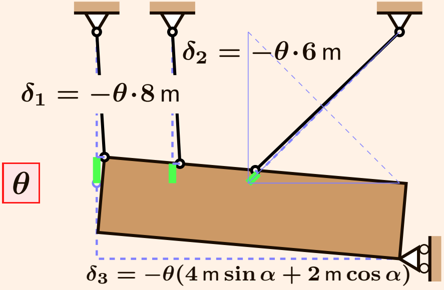


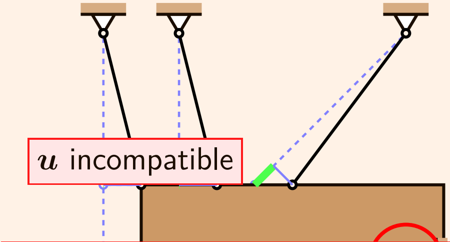
1
2
3
α
4 m
2 m
2 m
+
2 m
2 m
4 m
θ
$\delta_2 = -\theta \cdot 6\,\mathrm{m}$
$\delta_1 = -\theta \cdot 8\,\mathrm{m}$
$\delta_3 = -\theta(4\,\mathrm{m}\sin\alpha + 2\,\mathrm{m}\cos\alpha)$
v
$\delta_2 = v$
$\delta_1 = v$
$\delta_3 = v\sin\alpha$

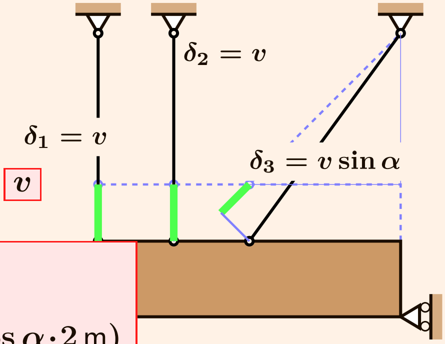


1
2
3
4 m
α
2 m
2 m
+
2 m
2 m
4 m
$\delta_2 = -\theta \cdot 6$ m
$\delta_1 = -\theta \cdot 8$ m
θ
$\delta_3 = -\theta(4\,\mathrm{m}\sin\alpha + 2\,\mathrm{m}\cos\alpha)$
u incompatible
$\delta_2 = v$
$\delta_1 = v$
$\delta_3 = v\sin\alpha$
v

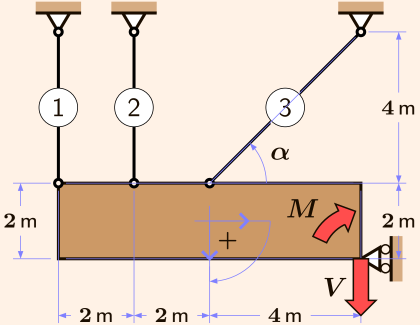


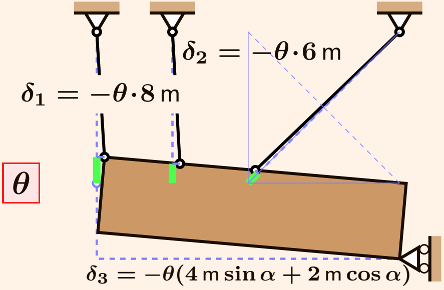




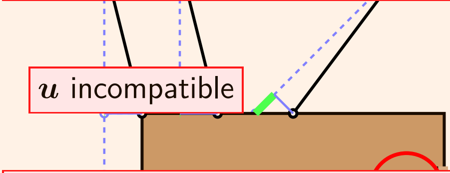


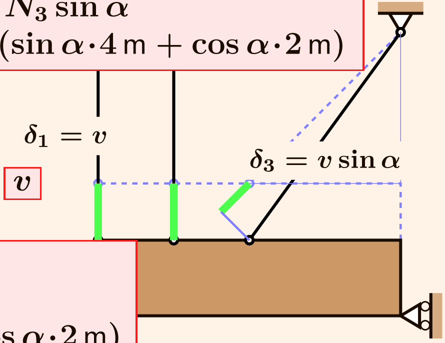
$$\delta_1 = v - \theta \cdot 8\,\mathrm{m}$$
$$\delta_2 = v - \theta \cdot 6\,\mathrm{m}$$
$$\delta_3 = v\sin\alpha - \theta(\sin\alpha \cdot 4\,\mathrm{m} + \cos\alpha \cdot 2\,\mathrm{m})$$



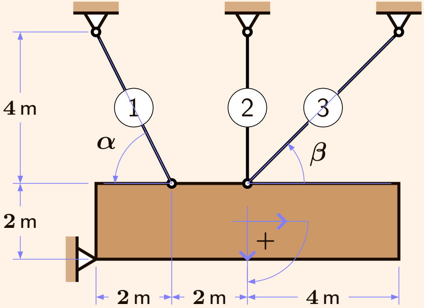


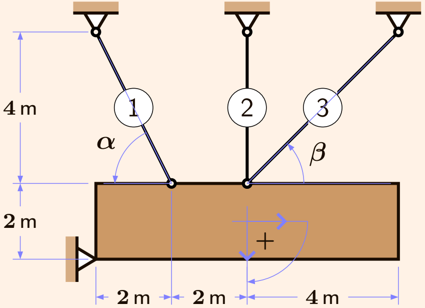
$$V = N_1 + N_2 + N_3 \sin\alpha$$
$$M = -N_1 \cdot 8\,\mathrm{m} - N_2 \cdot 6\,\mathrm{m} - N_3(\sin\alpha \cdot 4\,\mathrm{m} + \cos\alpha \cdot 2\,\mathrm{m})$$



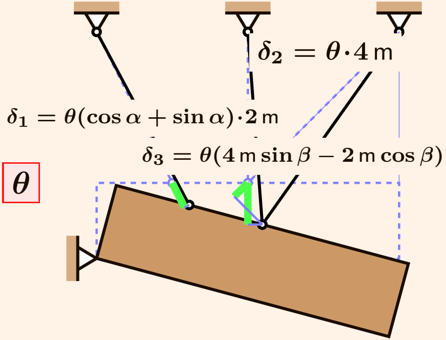


$$\delta_1 = v - \theta \cdot 8\,\mathrm{m}$$
$$\delta_2 = v - \theta \cdot 6\,\mathrm{m}$$
$$\delta_3 = v\sin\alpha - \theta(\sin\alpha \cdot 4\,\mathrm{m} + \cos\alpha \cdot 2\,\mathrm{m})$$

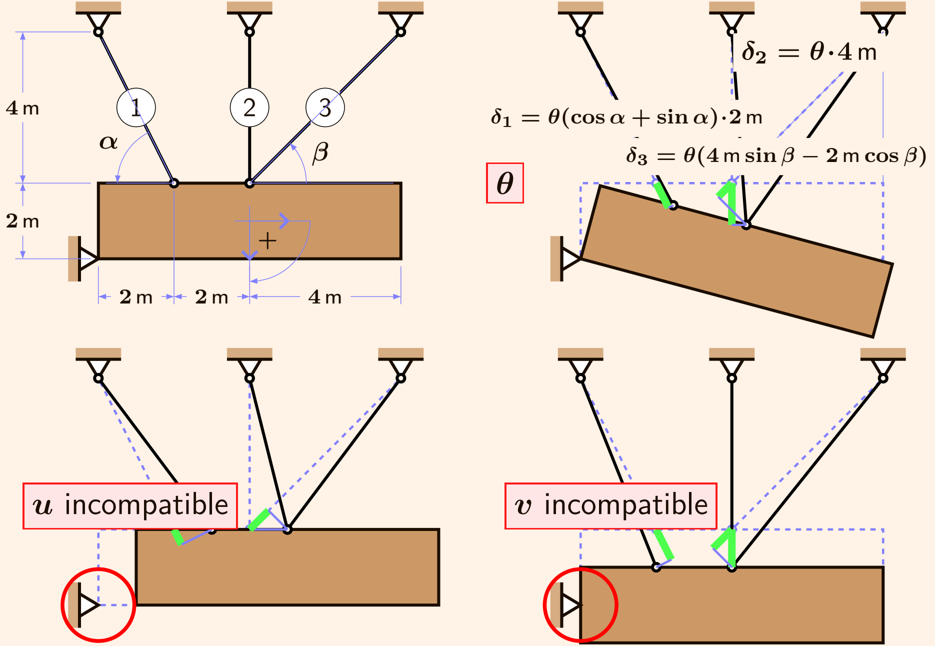

4 m
1
2
3
α
β
2 m
+
2 m
2 m
4 m

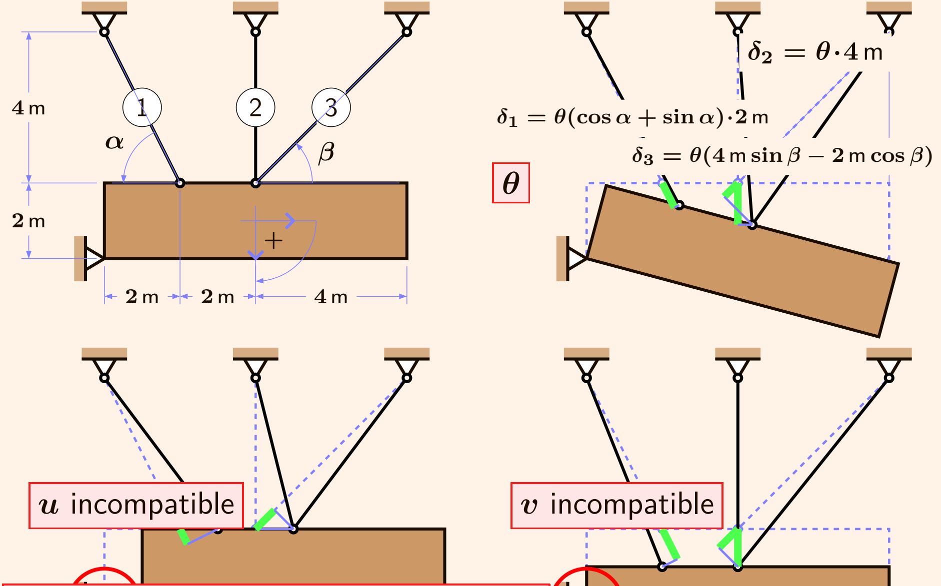


4 m
2 m
1
2
3
α
β
+
2 m
2 m
4 m

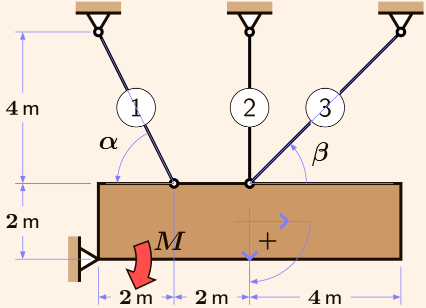


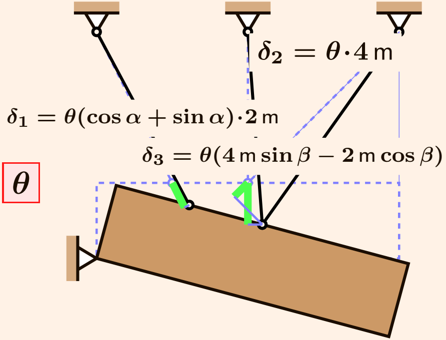
$\delta_2 = \theta \cdot 4\,\mathrm{m}$
$\delta_1 = \theta(\cos\alpha + \sin\alpha) \cdot 2\,\mathrm{m}$
$\delta_3 = \theta(4\,\mathrm{m}\sin\beta - 2\,\mathrm{m}\cos\beta)$
θ


4 m
2 m
1
2
3
α
β
2 m
2 m
4 m
+
$\delta_2 = \theta \cdot 4\,\text{m}$
$\delta_1 = \theta(\cos\alpha + \sin\alpha)\cdot 2\,\text{m}$
$\delta_3 = \theta(4\,\text{m}\sin\beta - 2\,\text{m}\cos\beta)$
θ
u incompatible
v incompatible

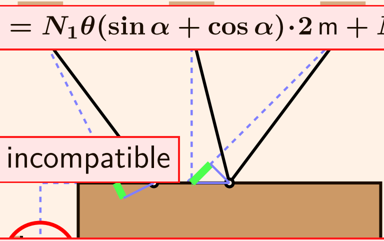


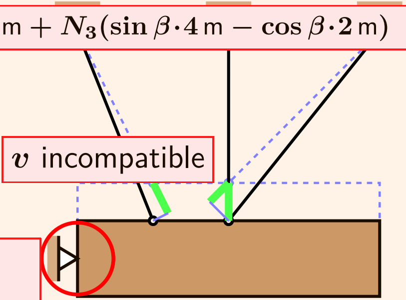
$$\delta_1 = \theta(\sin\alpha + \cos\alpha)\cdot 2\,\mathrm{m}$$
$$\delta_2 = \theta\cdot 4\,\mathrm{m}$$
$$\delta_3 = \theta(\sin\beta\cdot 4\,\mathrm{m} - \cos\beta\cdot 2\,\mathrm{m})$$



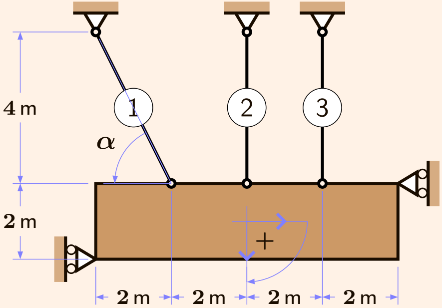


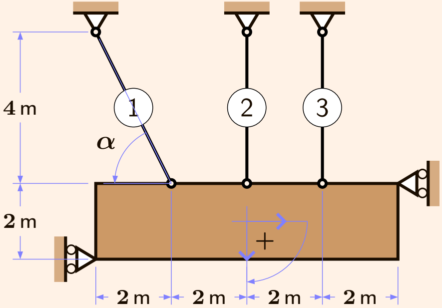
$$M = N_1\theta(\sin\alpha + \cos\alpha)\cdot 2\,\mathrm{m} + N_2\cdot 4\,\mathrm{m} + N_3(\sin\beta\cdot 4\,\mathrm{m} - \cos\beta\cdot 2\,\mathrm{m})$$

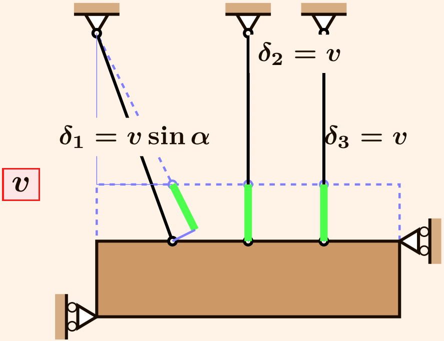


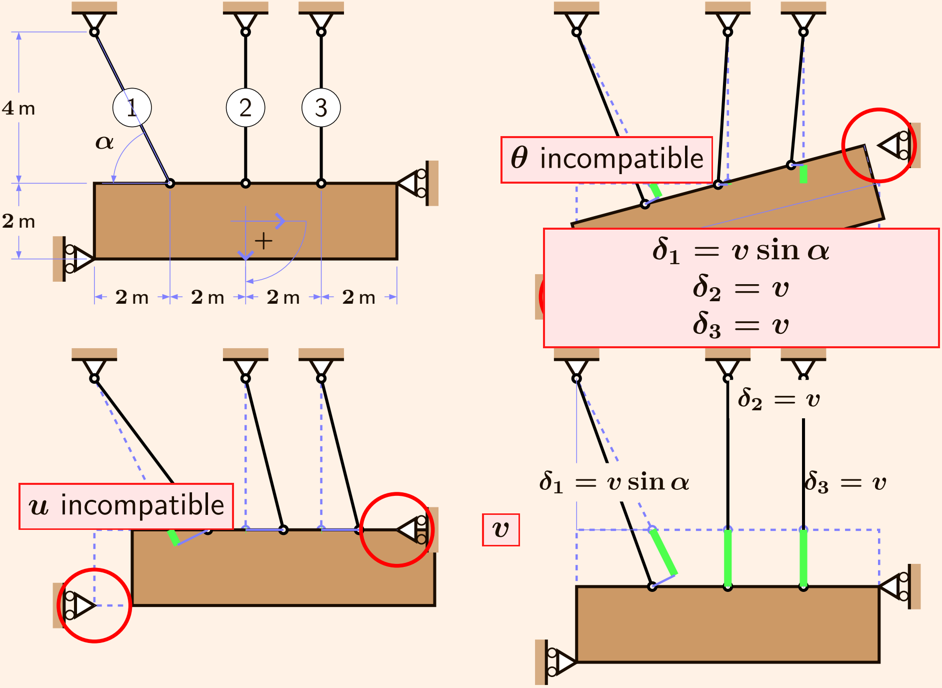


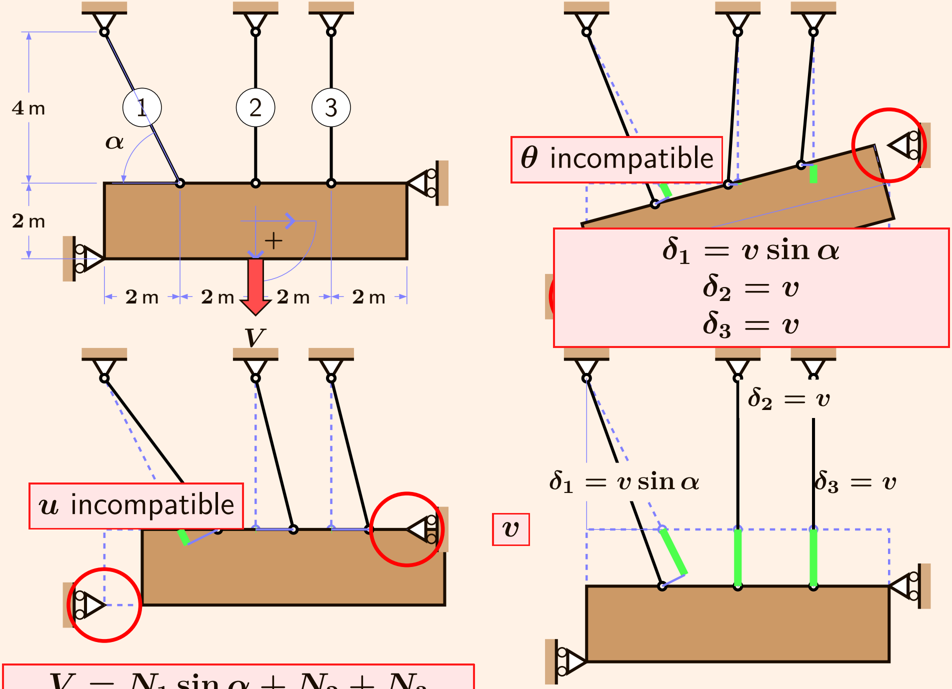
$$\delta_1 = \theta(\sin\alpha + \cos\alpha)\cdot 2\,\mathrm{m}$$
$$\delta_2 = \theta\cdot 4\,\mathrm{m}$$
$$\delta_3 = \theta(\sin\beta\cdot 4\,\mathrm{m} - \cos\beta\cdot 2\,\mathrm{m})$$


4 m
2 m
1
2
3
α
+
2 m
2 m
2 m
2 m


4 m
2 m
α
1
2
3
+
2 m
2 m
2 m
2 m

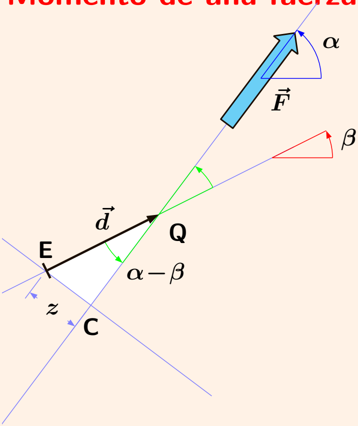

$\delta_2 = v$
$\delta_1 = v \sin \alpha$
$\delta_3 = v$
v


4 m
2 m
1
2
3
α
+
2 m
2 m
2 m
2 m
θ incompatible
δ1 = v sin α
δ2 = v
δ3 = v
u incompatible
v
δ2 = v
δ1 = v sin α
δ3 = v



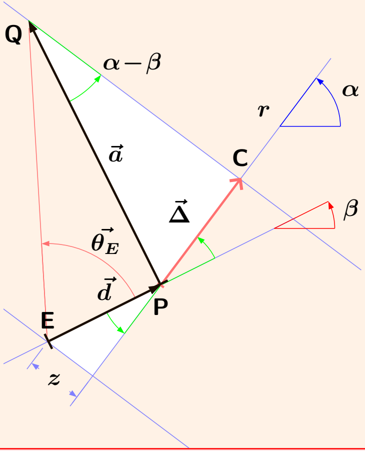
$$V = N_1 \sin\alpha + N_2 + N_3$$

Momento de una fuerza



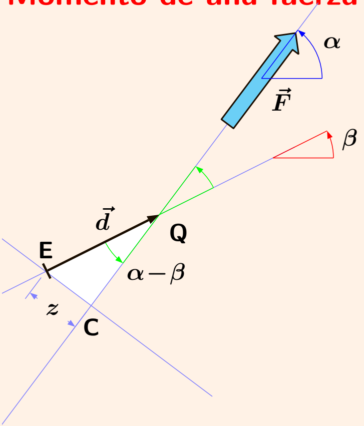
$$\vec{M}_E(\vec{F}) = \vec{F} \times \vec{d}$$
$$M_E = F \cdot d \cdot \sin(\alpha - \beta)$$
$$M_E = F \cdot z$$

Desplazamiento de un giro



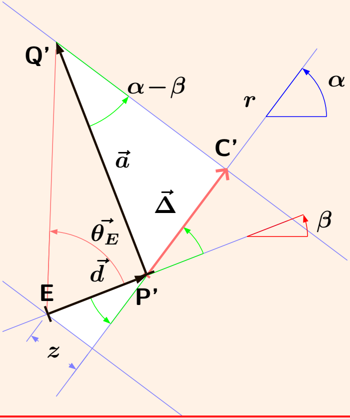
$$\vec{a}(\vec{\theta_E}) = \vec{d} \times \vec{\theta_E};\ a = \theta_E \cdot d$$
$$\Delta = \theta_E \cdot d \cdot \sin(\alpha - \beta)$$
$$\Delta = \theta_E \cdot z$$

Momento de una fuerza



$$\vec{M}_E(\vec{F}) = \vec{F} \times \vec{d}$$
$$M_E = F \cdot d \cdot \sin(\alpha - \beta)$$
$$M_E = F \cdot z$$

Desplazamiento de un giro



$$\vec{a}(\vec{\theta_E}) = \vec{d} \times \vec{\theta_E};\ a = \theta_E \cdot d$$
$$\Delta = \theta_E \cdot d \cdot \sin(\alpha - \beta)$$
$$\Delta = \theta_E \cdot z\ ;\ \vec{\Delta} = \vec{z} \times \vec{\theta_E}$$

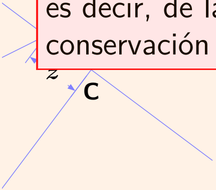
Relaciones mecánicas y cinemáticas

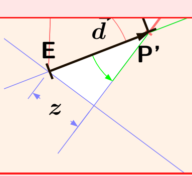
M **iro**

La relación de "contravarianza" entre la pareja momento y giro y la pareja fuerza y desplazamiento, surge de la necesidad de que el trabajo de una fuerza resulte igual al trabajo de "su" momento:

$$\text{trabajo} = \boldsymbol{M_E} \cdot \boldsymbol{\theta_E} = (\boldsymbol{F} \cdot \boldsymbol{z}) \cdot \boldsymbol{\theta_E} = \boldsymbol{F} \cdot (\boldsymbol{z} \cdot \boldsymbol{\theta_E}) = \boldsymbol{F} \cdot \boldsymbol{\Delta}$$

es decir, de la primera ley de la termodinámica, la de conservación de la energía.





$$\vec{M}_E(\vec{F}) = \vec{F} \times \vec{d}$$
$$M_E = F \cdot d \cdot \sin(\alpha - \beta)$$
$$M_E = F \cdot z$$

$$\vec{a}(\vec{\theta_E}) = \vec{d} \times \vec{\theta_E};\ a = \theta_E \cdot d$$
$$\Delta = \theta_E \cdot d \cdot \sin(\alpha - \beta)$$
$$\Delta = \theta_E \cdot z\ ;\ \vec{\Delta} = \vec{z} \times \vec{\theta_E}$$

Sólido deformable: compatibilidad y equilibrio (3rd ed)

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