Diagramas de esfuerzos

(Funiculares como diagramas)

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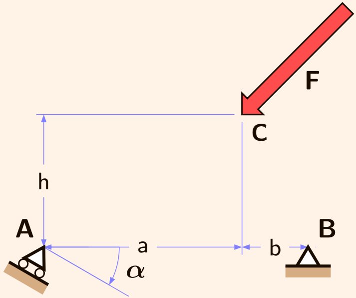
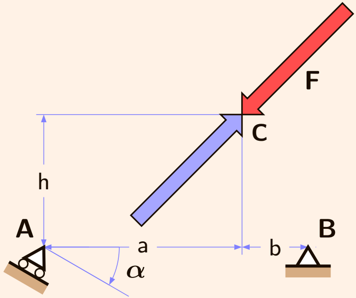
Ondara/Madrid, 27 de febrero de 2020.

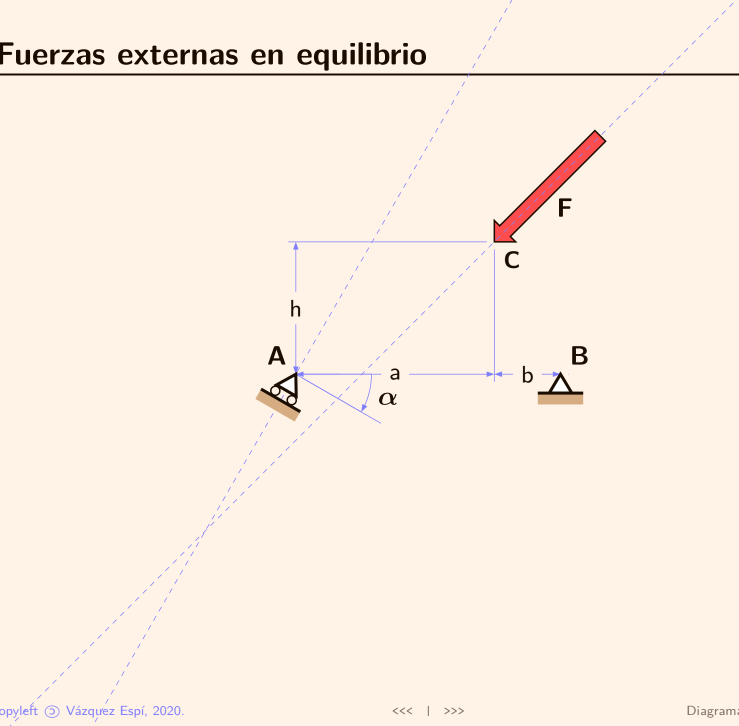
Fuerzas externas en equilibrio



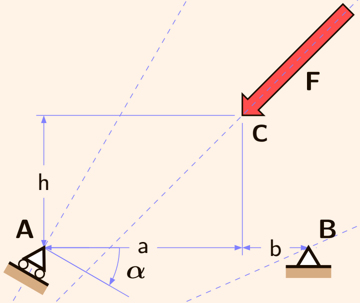
Fuerzas externas en equilibrio



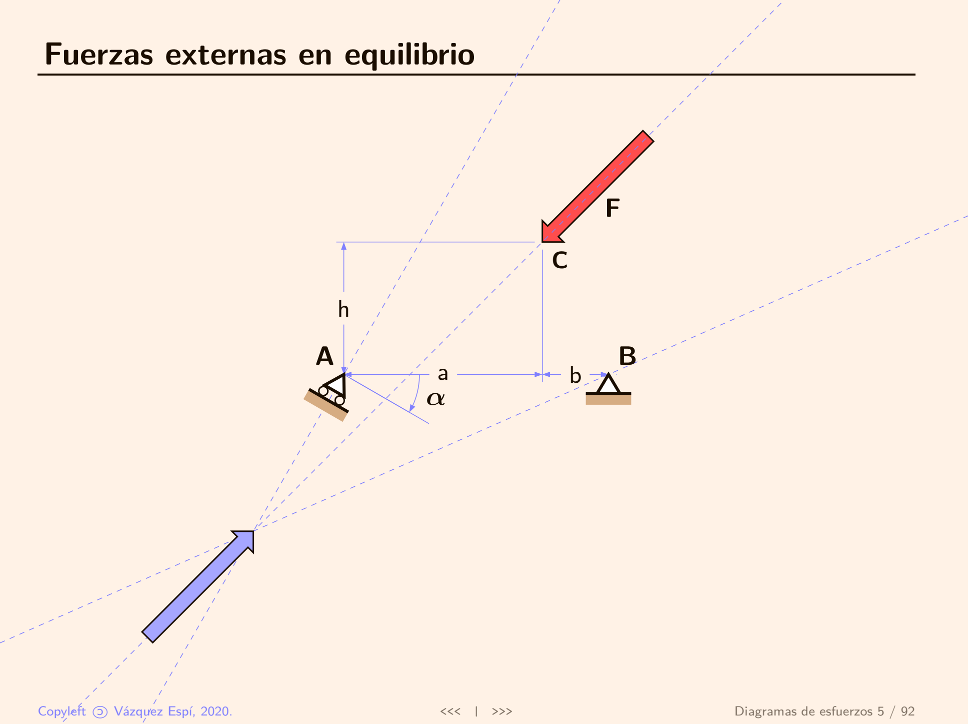
Fuerzas externas en equilibrio



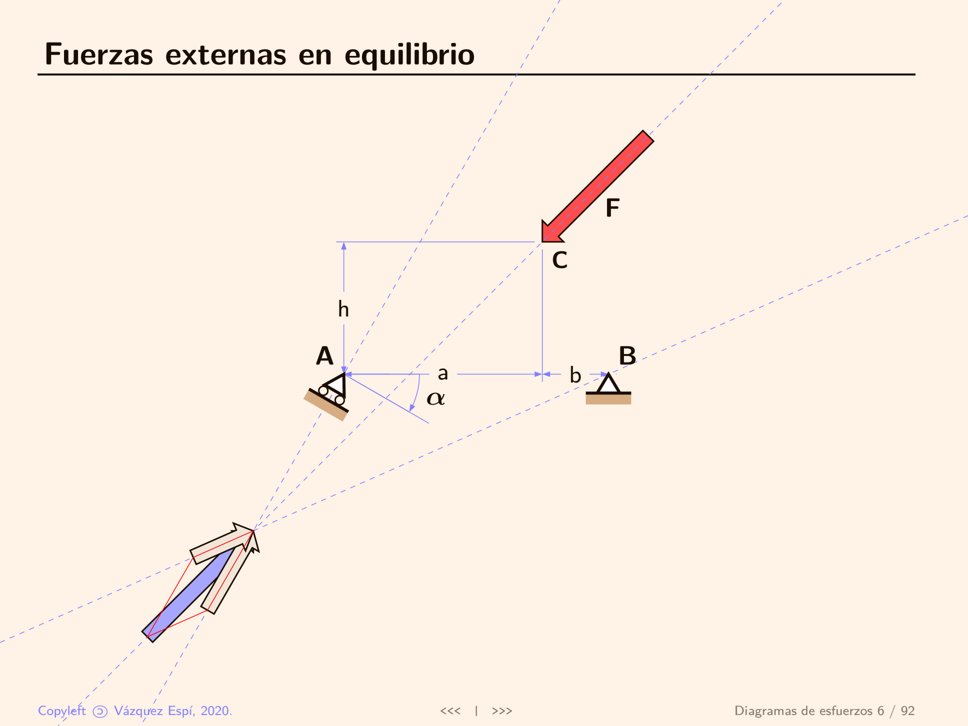
Fuerzas externas en equilibrio



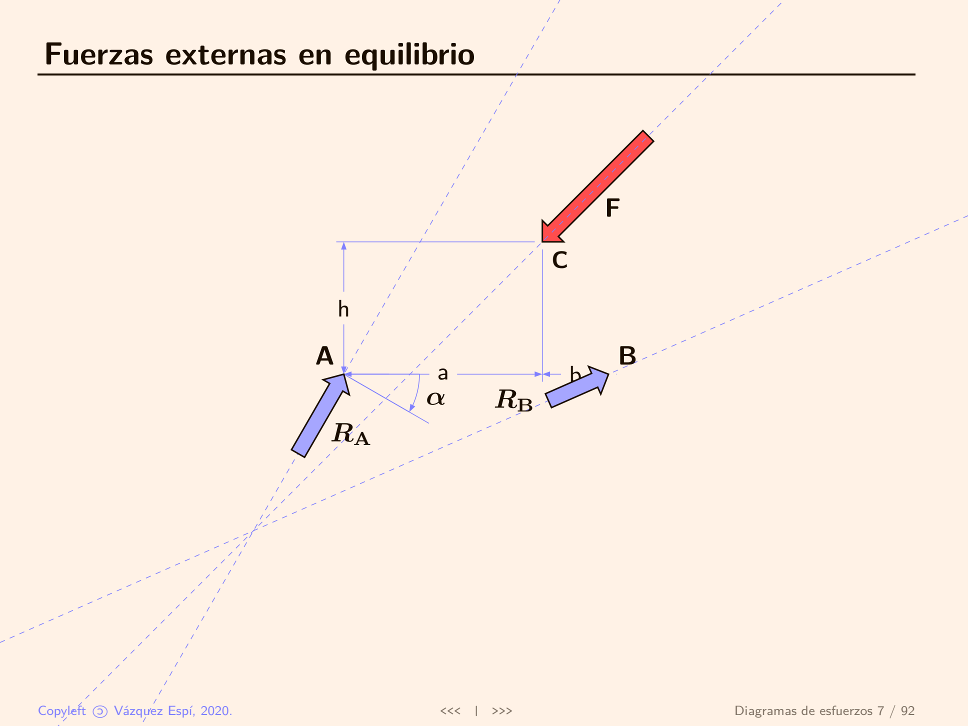
Fuerzas externas en equilibrio



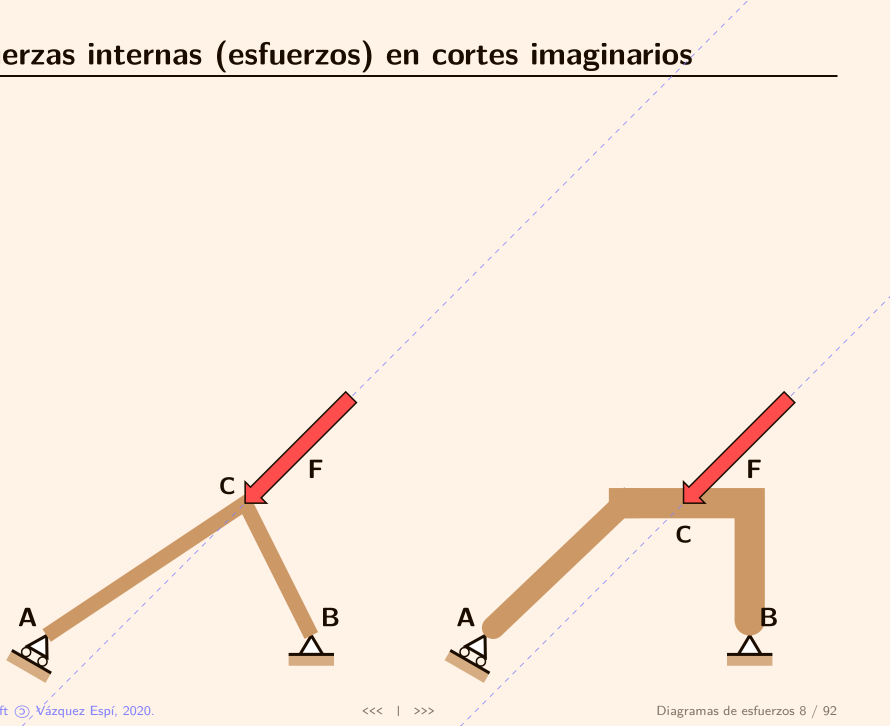
Fuerzas externas en equilibrio



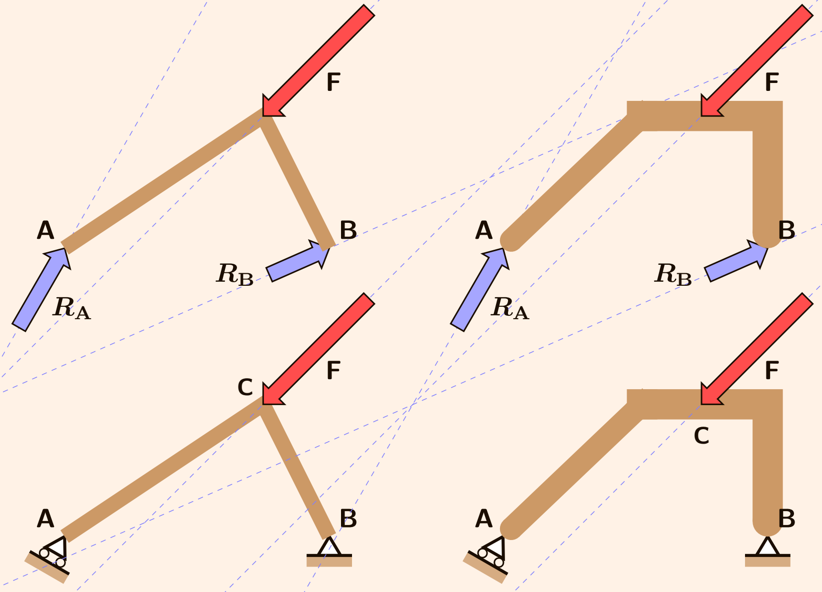
Fuerzas externas en equilibrio



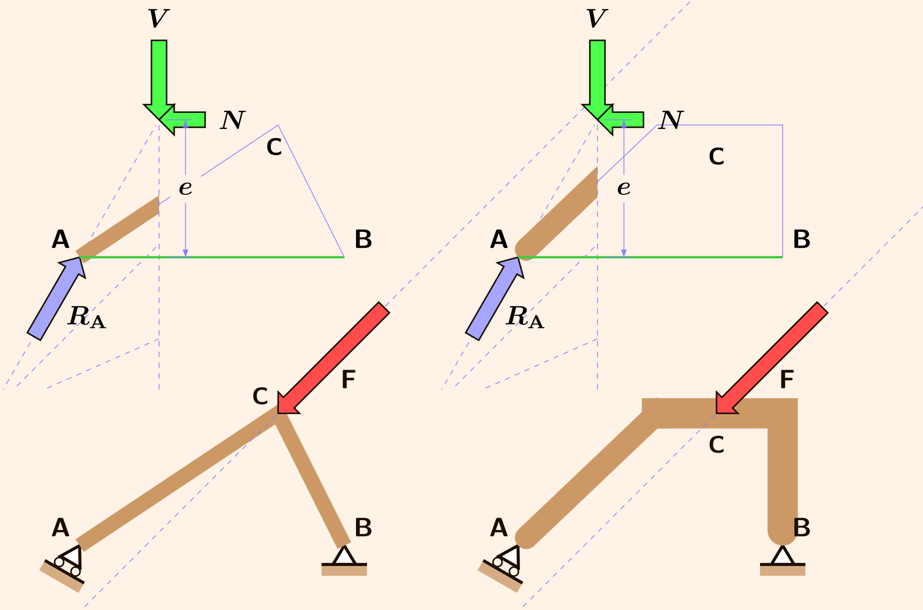
Fuerzas internas (esfuerzos) en cortes imaginarios



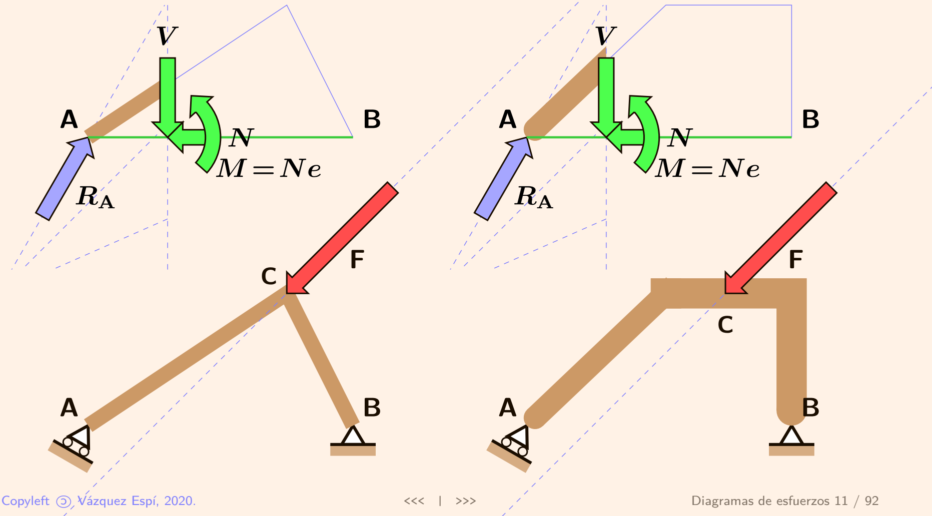
Fuerzas internas (esfuerzos) en cortes imaginarios



Fuerzas internas (esfuerzos) en cortes imaginarios



Fuerzas internas (esfuerzos) en cortes imaginarios



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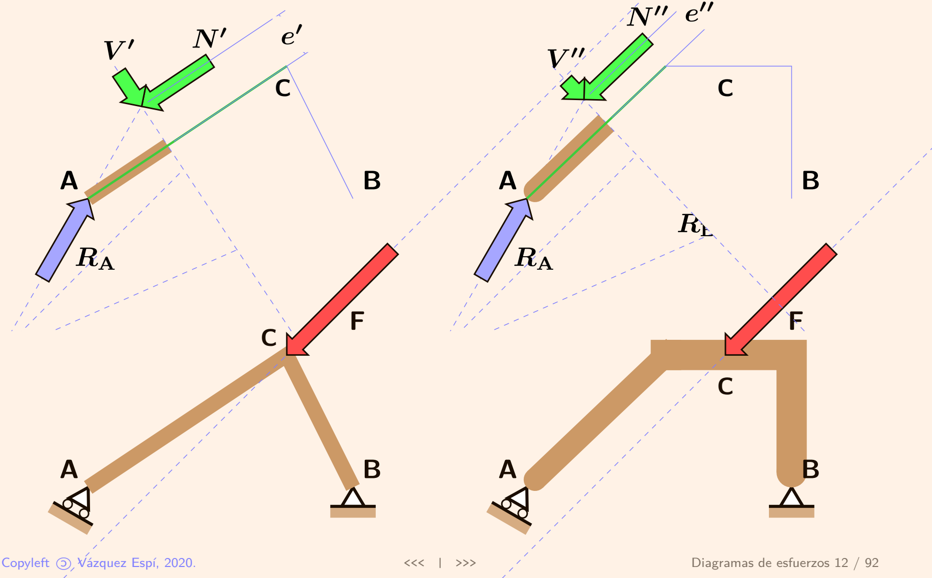
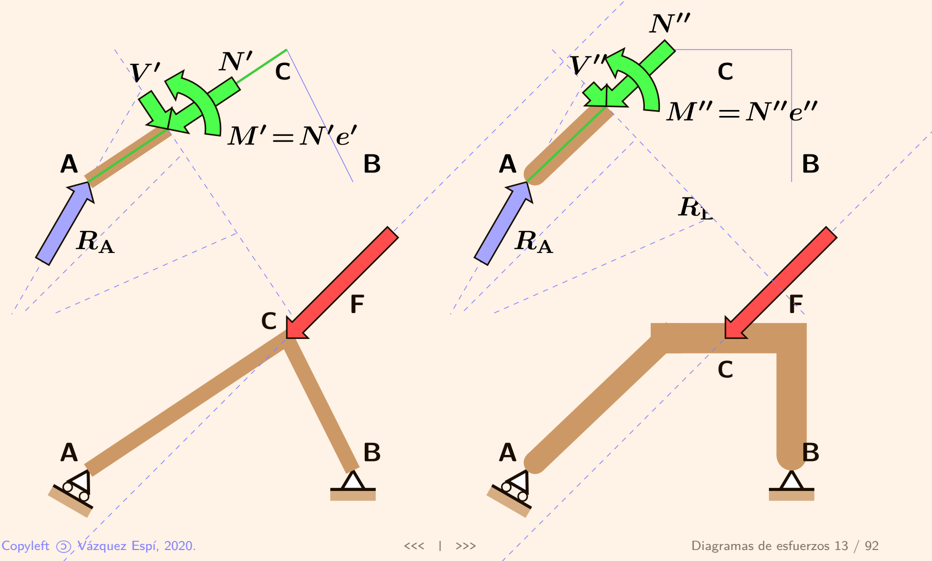
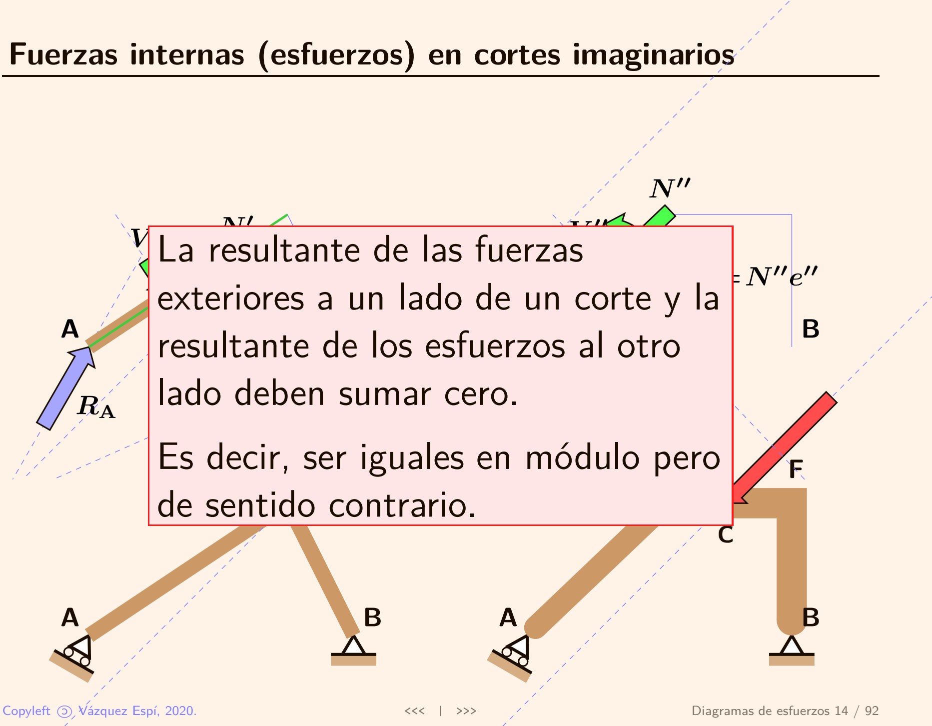


Fuerzas internas (esfuerzos) en cortes imaginarios



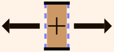
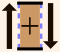
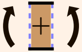
Fuerzas internas (esfuerzos) en cortes imaginarios



La resultante de las fuerzas exteriores a un lado de un corte y la resultante de los esfuerzos al otro lado deben sumar cero.

Es decir, ser iguales en módulo pero de sentido contrario.

Solicitaciones y esfuerzos

	Esfuerzos		
Solicitación	Longitudinal Normal Axil	Transversal Cortante	Par Flector Momento flector
Tracción simple	N	—	—
Flexión simple	—	V	M
Flexión compuesta Tracción compuesta Compresión compuesta	N	V	M
Compresión simple	N	—	M
Cizalladura	—	V	—
Flexión pura	—	—	M
	σ constante 	τ 	σ variable 

Formulación analítica

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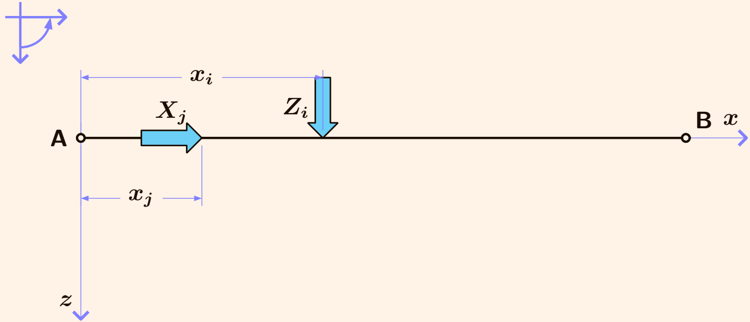

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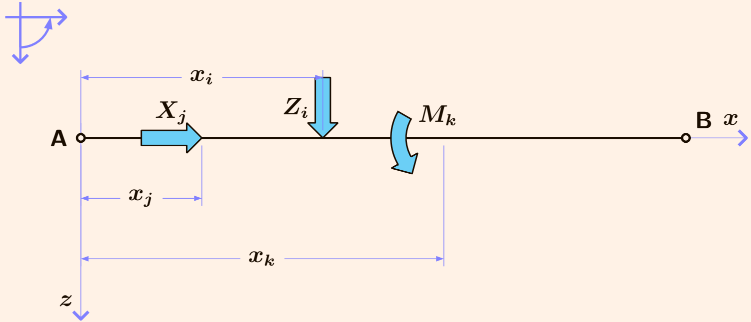

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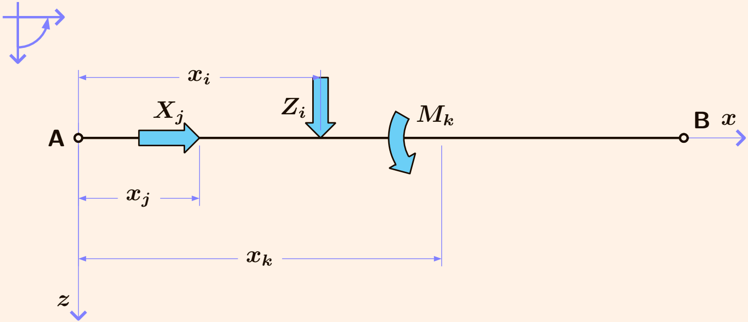
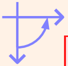


Problema de Maxwell: Son conocidas las fuerzas externas (acciones y reacciones) y **están en equilibrio**, es decir, que considerando todas ellas entre **A** y **B**:

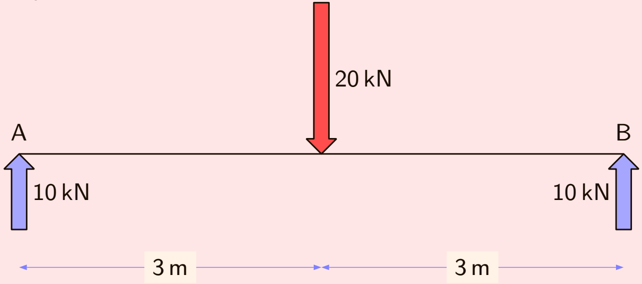
$$\sum X_j = 0 \qquad \sum Z_i = 0$$

$$\sum \vec{M}_A = 0 \quad \Rightarrow \quad \sum M_k - \sum Z_i \cdot x_i = 0$$

Formulación analítica



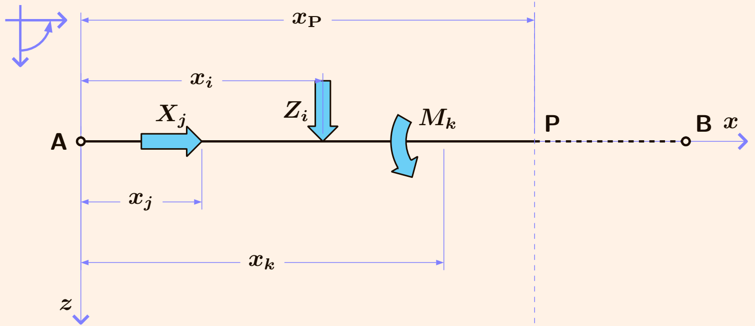
Ejemplo:



$$\sum \boldsymbol{X_j} : \mathbf{0} = \mathbf{0} \qquad \sum \boldsymbol{Z_i} : \mathbf{20}\,\mathrm{kN} - \mathbf{10}\,\mathrm{kN} - \mathbf{10}\,\mathrm{kN} = \mathbf{0}$$

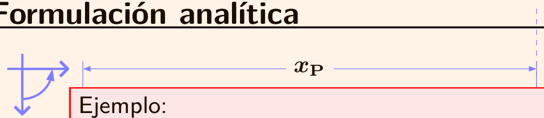
$$\sum \boldsymbol{M_{\mathrm{A}}} : -\mathbf{20}\,\mathrm{kN} \times \mathbf{3}\,\mathrm{m} + \mathbf{10}\,\mathrm{kN} \times \mathbf{6}\,\mathrm{m} = \mathbf{0}$$

Formulación analítica



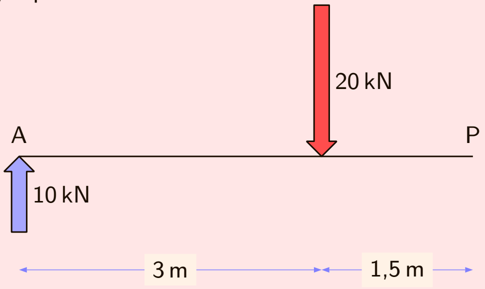
Equilibrio en un trozo: suma entre **A** y un punto cualquiera **P**. En general, las fuerzas externas del trozo estarán desequilibradas.

Formulación analítica



A

Ejemplo:

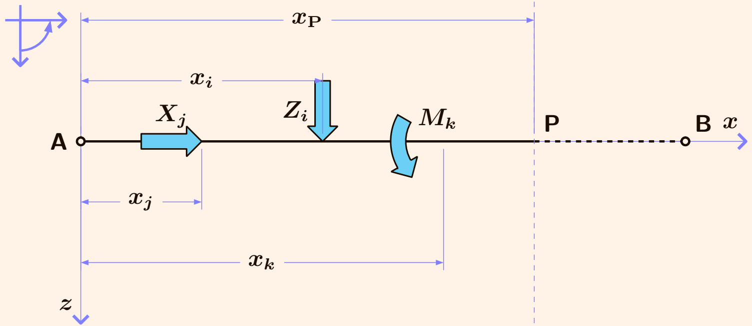


$$\sum X_j : \mathbf{0} = \mathbf{0} \qquad \sum Z_i : \mathbf{20}\,\text{kN} - \mathbf{10}\,\text{kN} = \mathbf{10}\,\text{kN} \neq \mathbf{0}$$

$$\sum M_{\mathbf{A}} : -\mathbf{20}\,\text{kN} \times \mathbf{3}\,\text{m} = -\mathbf{60}\,\text{mkN} \neq \mathbf{0}$$

Equilib ... **P**. En general ... s.

Formulación analítica

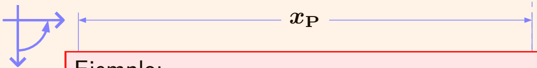


Desequilibrio de las fuerzas externas, es decir, resultantes de las fuerzas externas en el trozo **AP**:

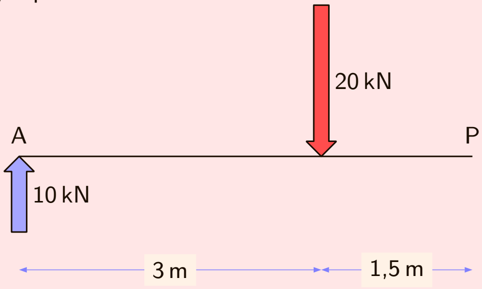
$$R_X(x_{\mathrm{P}}) = \sum_{x_j \le x_{\mathrm{P}}} X_j \qquad R_Z(x_{\mathrm{P}}) = \sum_{x_i \le x_{\mathrm{P}}} Z_i$$

$$R_M(x_{\mathrm{P}}) = \sum_{x_i \le x_{\mathrm{P}}} Z_i \cdot (x_{\mathrm{P}} - x_i) + \sum_{x_k \le x_{\mathrm{P}}} M_k$$

Formulación analítica



Ejemplo:



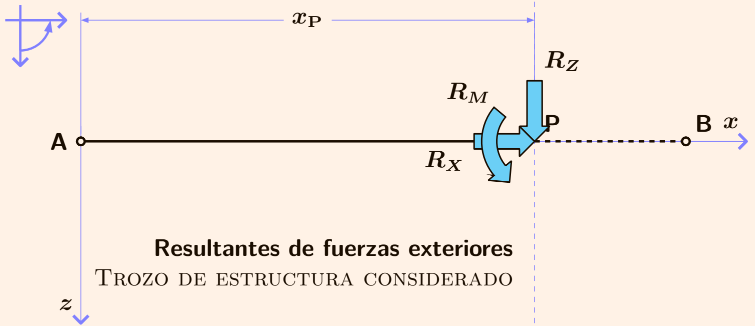
$$R_X(\mathrm{P}) = 0 \qquad R_Z(\mathrm{P}) = 20\,\mathrm{kN} - 10\,\mathrm{kN} = 10\,\mathrm{kN} \neq 0$$

$$R_M(\mathrm{P}) = 20\,\mathrm{kN} \times 1{,}5\,\mathrm{m} - 10\,\mathrm{kN} \times 4{,}5\,\mathrm{m} = -15\,\mathrm{mkN} \neq 0$$

Desequ … as fuerzas ext …

$$R_M(x_\mathrm{P}) = \sum_{x_i \leq x_\mathrm{P}} Z_i \cdot (x_\mathrm{P} - x_i) + \sum_{x_k \leq x_\mathrm{P}} M_k$$

Formulación analítica



Desequilibrio de las fuerzas externas, es decir, resultantes de las fuerzas externas <u>en el trozo **AP**</u>:

$$R_X(x_{\mathrm{P}}) = \sum_{x_j \leq x_{\mathrm{P}}} X_j \qquad R_Z(x_{\mathrm{P}}) = \sum_{x_i \leq x_{\mathrm{P}}} Z_i$$

$$R_M(x_{\mathrm{P}}) = \sum_{x_i \leq x_{\mathrm{P}}} Z_i \cdot (x_{\mathrm{P}} - x_i) + \sum_{x_k \leq x_{\mathrm{P}}} M_k$$

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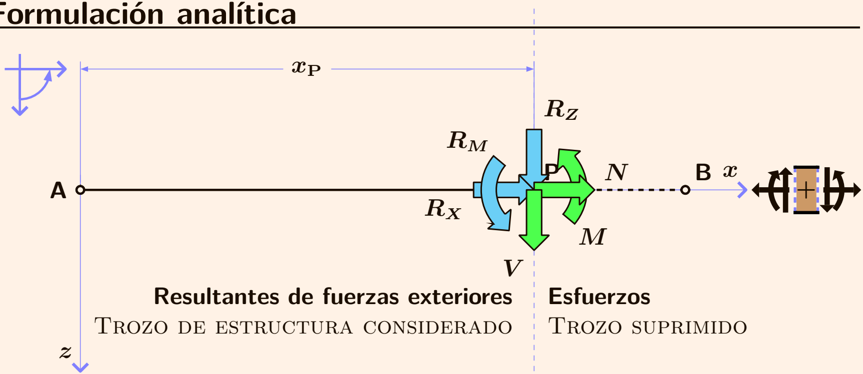
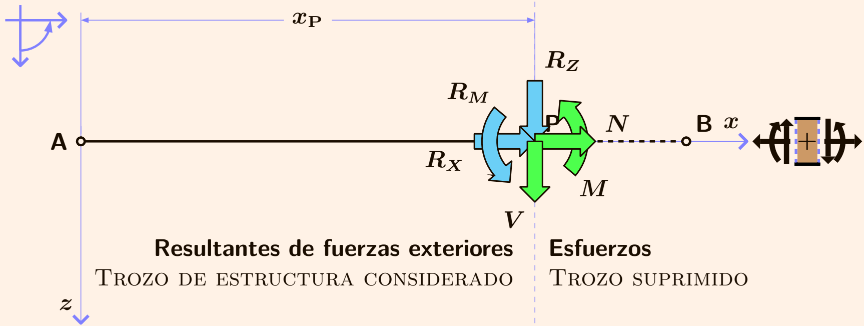


Desequilibrio de las fuerzas externas, es decir, resultantes de las fuerzas externas en el trozo **AP**:

$$R_X(x_{\mathrm{P}}) = \sum_{x_j \leq x_{\mathrm{P}}} X_j \qquad R_Z(x_{\mathrm{P}}) = \sum_{x_i \leq x_{\mathrm{P}}} Z_i$$

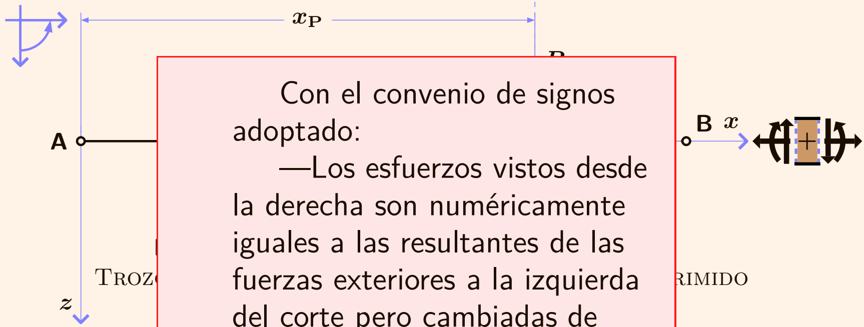
$$R_M(x_{\mathrm{P}}) = \sum_{x_i \leq x_{\mathrm{P}}} Z_i \cdot (x_{\mathrm{P}} - x_i) + \sum_{x_k \leq x_{\mathrm{P}}} M_k$$

Formulación analítica



Equilibrio del trozo (entre fuerzas externas e internas):

$$\left\{\begin{array}{l} R_X(x_P) + N(x_P) = 0 \\ R_Z(x_P) + V(x_P) = 0 \\ R_M(x_P) + M(x_P) = 0 \end{array}\right. \Rightarrow \left\{\begin{array}{l} N(x_P) = -R_X(x_P) \\ V(x_P) = -R_Z(x_P) \\ M(x_P) = -R_M(x_P) \end{array}\right.$$



Con el convenio de signos adoptado:

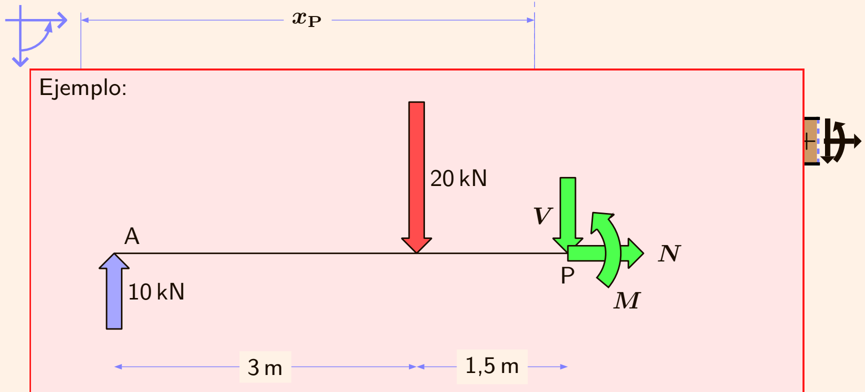
—Los esfuerzos vistos desde la derecha son numéricamente iguales a las resultantes de las fuerzas exteriores a la izquierda del corte pero cambiadas de signo.

—O, lo que es lo mismo, a las resultantes de las fuerzas exteriores a la derecha.

Equilibrio del …):

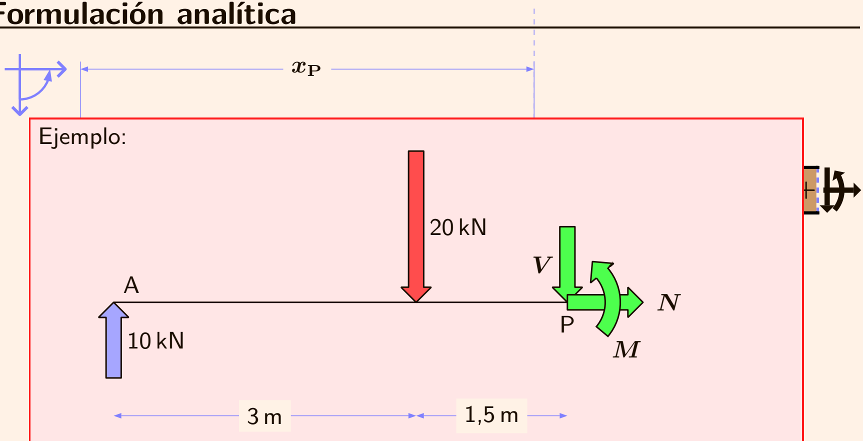
$$\left\{\begin{array}{l} R_X(x_P) \ldots \\ R_Z(x_P) \ldots \\ R_M(x_P) + M(x_P) = 0 \end{array}\right. \qquad \ldots R_X(x_P) \quad \ldots R_Z(x_P) \quad \ldots M(x_P) = -R_M(x_P)$$

Formulación analítica



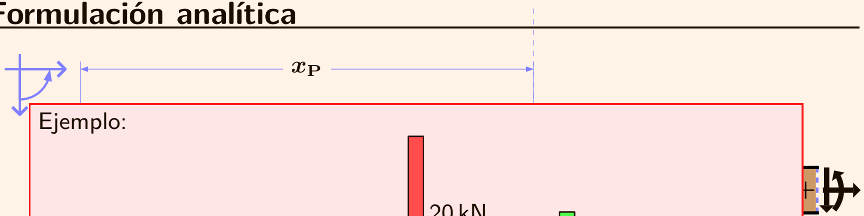
$$\sum X : \quad N = 0$$

Formulación analítica

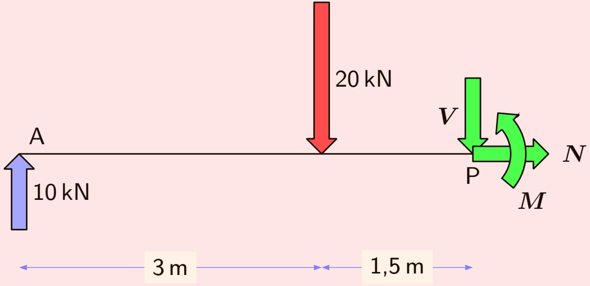


$$\sum Z: \quad -10\,\text{kN} + 20\,\text{kN} + V = 0 \quad V = -10\,\text{kN}$$

Formulación analítica



Ejemplo:

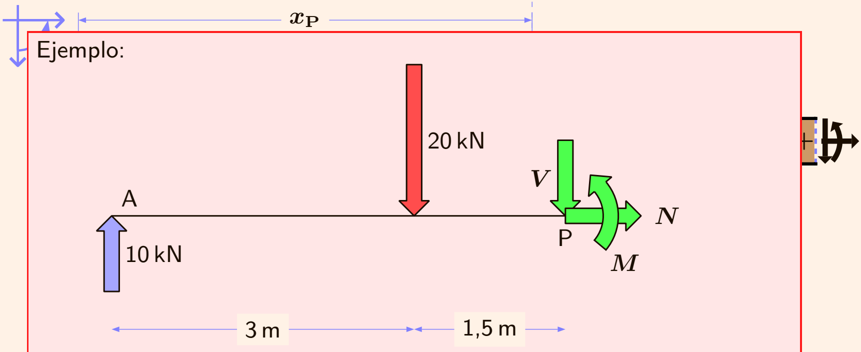


$$\sum M_A : \quad -20\,\text{kN} \times 3\,\text{m} - V \times 4{,}5\,\text{m} + M = 0 \quad \Rightarrow$$

$$\Rightarrow \quad M = 15\,\text{mkN}$$

Formulación analítica

Ejemplo:

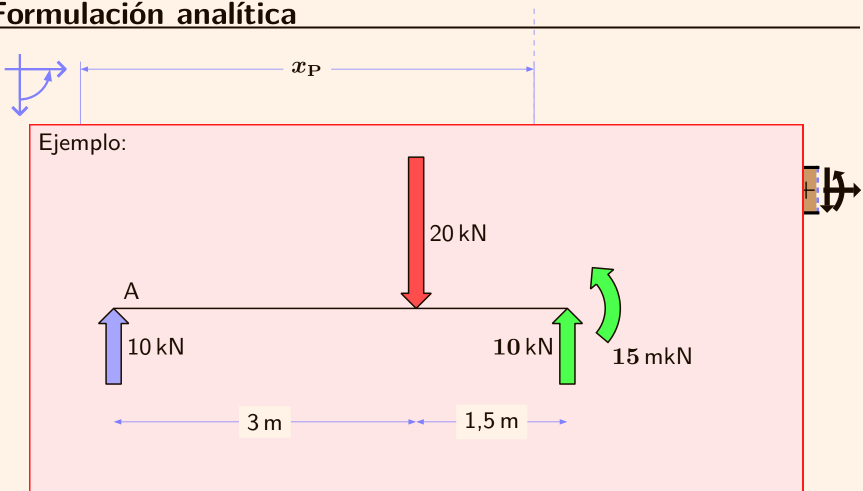


Alternativa canónica: equilibrio en **P**:

$$\sum M_P : \quad -10\,\text{kN} \times 4{,}5\,\text{m} + 20\,\text{kN} \times 1{,}5\,\text{m} + M = 0 \quad \Rightarrow$$

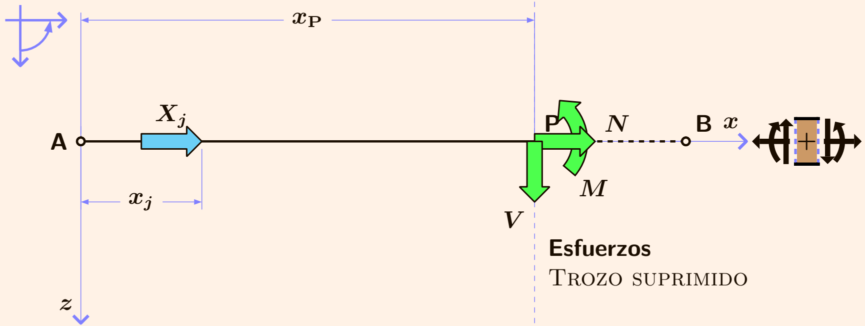
$$\Rightarrow \quad M = 15\,\text{mkN}$$

Formulación analítica



$$N = 0 \qquad V = -10\,\text{kN} \qquad M = 15\,\text{mkN}$$

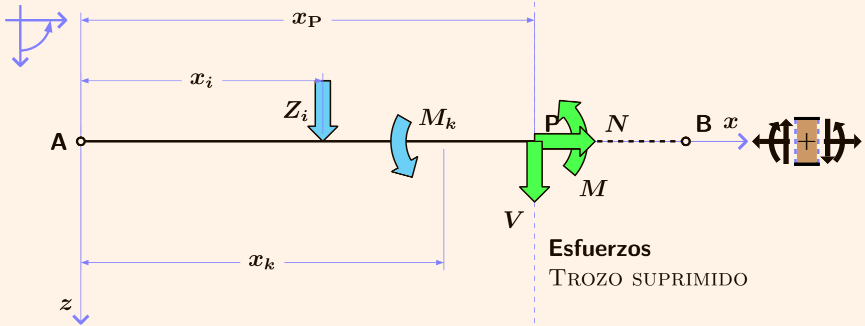
Formulación analítica



Ecuación para el diagrama de esfuerzos normales:

$$N(x_\mathrm{P}) = -\sum_{x_j \leq x_\mathrm{P}} X_j$$

Formulación analítica

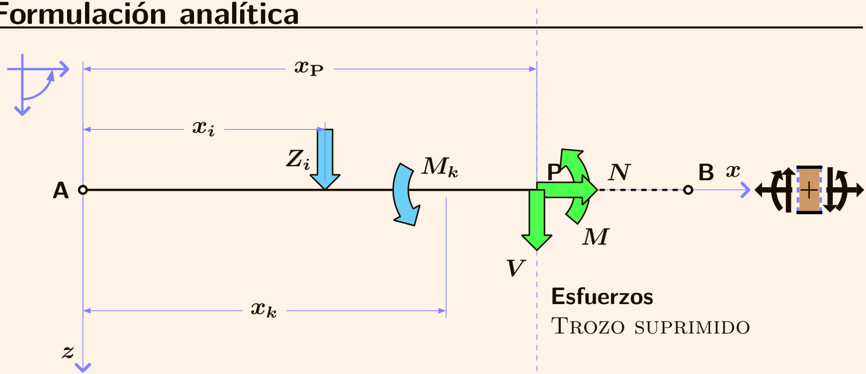


Ecuaciones para los diagramas de la flexión simple:

$$V(x_{\mathrm{P}}) = -\sum_{x_i \leq x_{\mathrm{P}}} Z_i$$

$$M(x_{\mathrm{P}}) = -\sum_{x_i \leq x_{\mathrm{P}}} Z_i \cdot (x_{\mathrm{P}} - x_i) - \sum_{x_k \leq x_{\mathrm{P}}} M_k$$

Formulación analítica



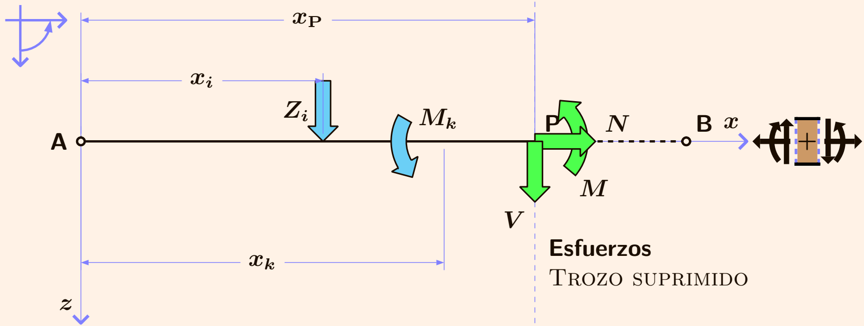
Ecuaciones para los diagramas de la flexión simple:

$$V(x_{\mathrm{P}}) = -\sum_{x_i \leq x_{\mathrm{P}}} Z_i$$

$$M(x_{\mathrm{P}}) = -\sum_{x_i \leq x_{\mathrm{P}}} Z_i \cdot (x_{\mathrm{P}} - x_i) - \sum_{x_k \leq x_{\mathrm{P}}} M_k$$

derivando. . .

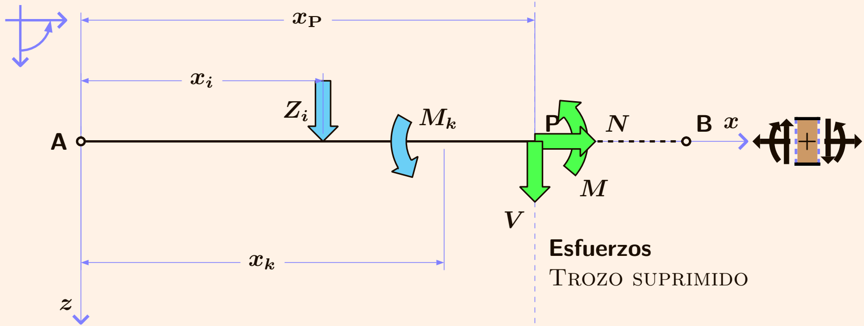
Formulación analítica



$$V(x_P) = -\sum_{x_i \leq x_P} Z_i$$

$$\frac{\partial M(x_P)}{\partial x_P} = -\sum_{x_i \leq x_P} Z_i = V(x_P)$$

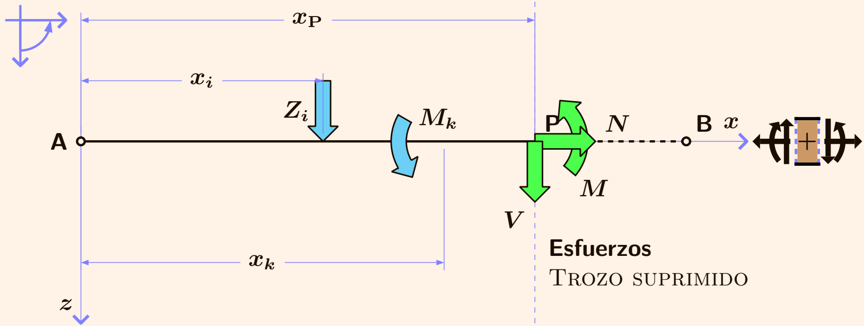
Formulación analítica



Ligazón con los funiculares:

$$M(x) = H \cdot z(x) \qquad \frac{\partial M(x)}{\partial x} = V(x) = H \cdot z'(x)$$

Formulación analítica

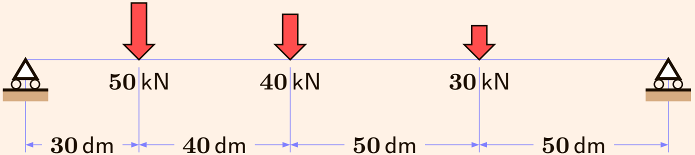


Generalización para cargas distribuidas:

$$M(x_c) = -\int_0^{x_c} p_z(x_c - x)\,\mathrm{d}x \qquad V(x_c) = -\int_0^{x_c} p_z\,\mathrm{d}x$$

$$\frac{\partial^2 M(x_c)}{\partial x_c^2} = \frac{\partial V(x_c)}{\partial x_c} = -p_z$$

Tres cargas entre dos apoyos



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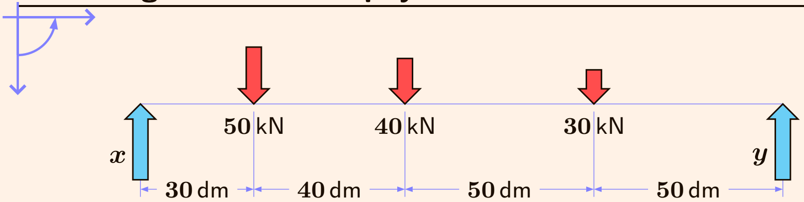

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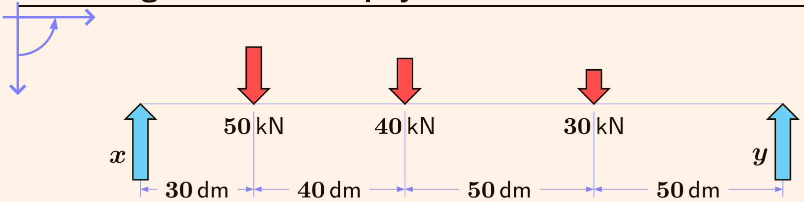
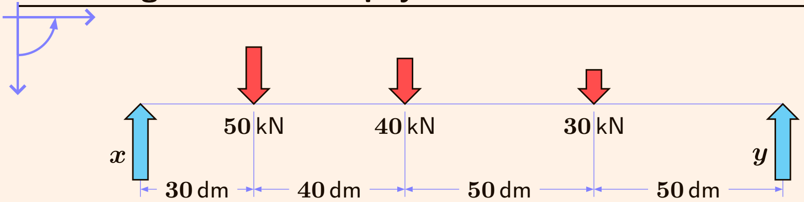


$$y \times 17\,\mathrm{m} - 50\,\mathrm{kN} \times 3\,\mathrm{m} - 40\,\mathrm{kN} \times 7\,\mathrm{m} - 30\,\mathrm{kN} \times 12\,\mathrm{m} = 0$$

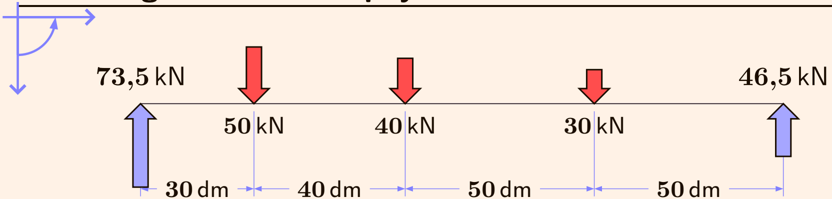
Tres cargas entre dos apoyos



$$y \times 17\,\mathrm{m} - 50\,\mathrm{kN} \times 3\,\mathrm{m} - 40\,\mathrm{kN} \times 7\,\mathrm{m} - 30\,\mathrm{kN} \times 12\,\mathrm{m} = 0$$

$$-x \times 17\,\mathrm{m} + 50\,\mathrm{kN} \times 14\,\mathrm{m} + 40\,\mathrm{kN} \times 10\,\mathrm{m} + 30\,\mathrm{kN} \times 5\,\mathrm{m} = 0$$

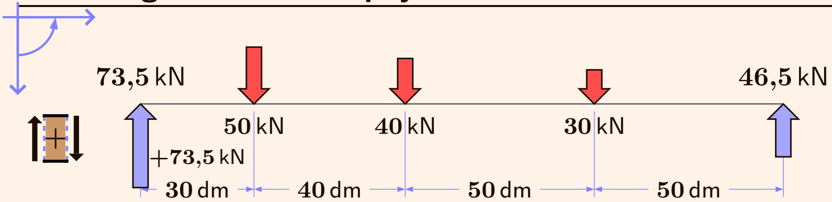
Tres cargas entre dos apoyos



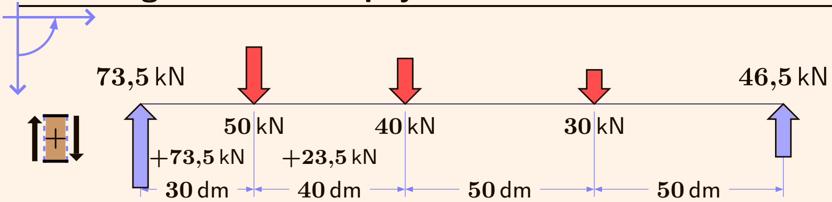
Comprobación:

$$73{,}5\,\text{kN} + 46{,}5\,\text{kN} = 120\,\text{kN} = 50\,\text{kN} + 40\,\text{kN} + 30\,\text{kN}$$

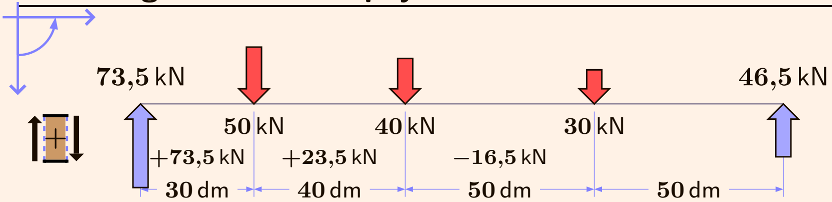
Tres cargas entre dos apoyos



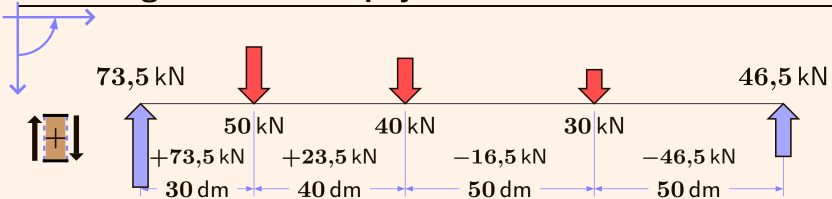
Tres cargas entre dos apoyos



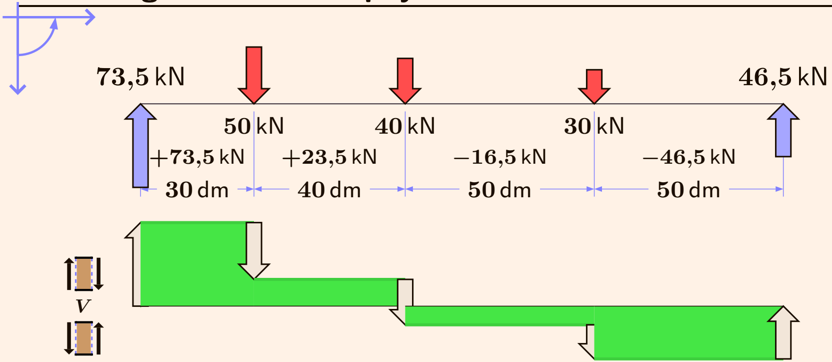
Tres cargas entre dos apoyos



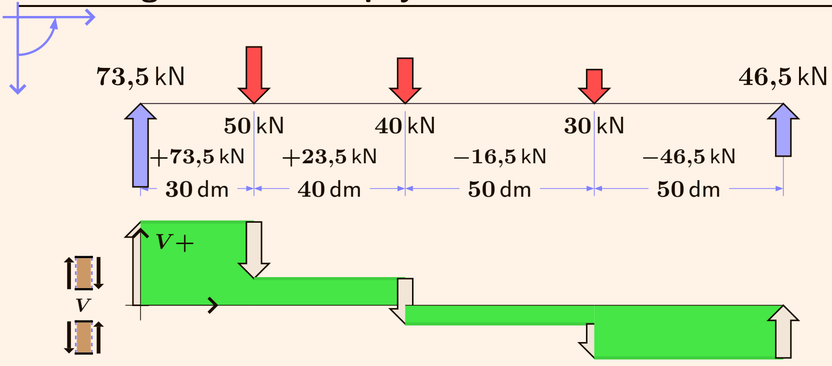
Tres cargas entre dos apoyos



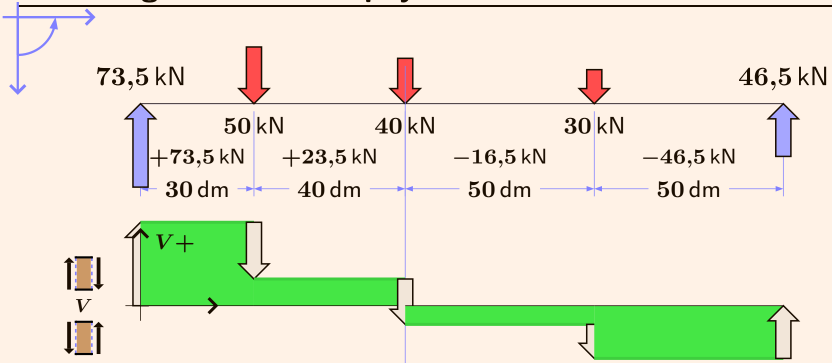
Tres cargas entre dos apoyos



Tres cargas entre dos apoyos



Tres cargas entre dos apoyos

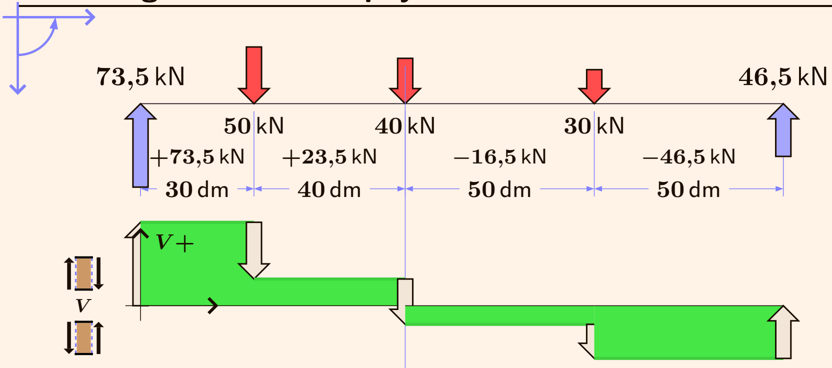


Cálculo de esfuerzos flectores:

$$\sum M_{\mathbf{7.izqd}} : -73{,}5\,\mathrm{kN} \times 7\,\mathrm{m} + 50\,\mathrm{kN} \times 4\,\mathrm{m} + M(7\,\mathrm{m}) = 0$$

$$M(7\,\mathrm{m}) = 314{,}5\,\mathrm{mkN}$$

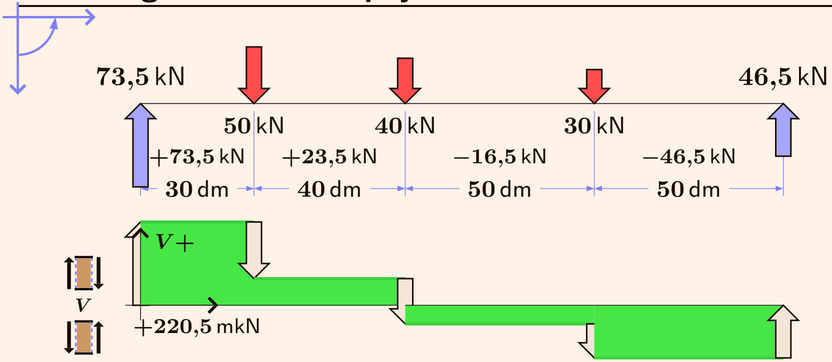
Tres cargas entre dos apoyos



Cálculo de esfuerzos flectores:

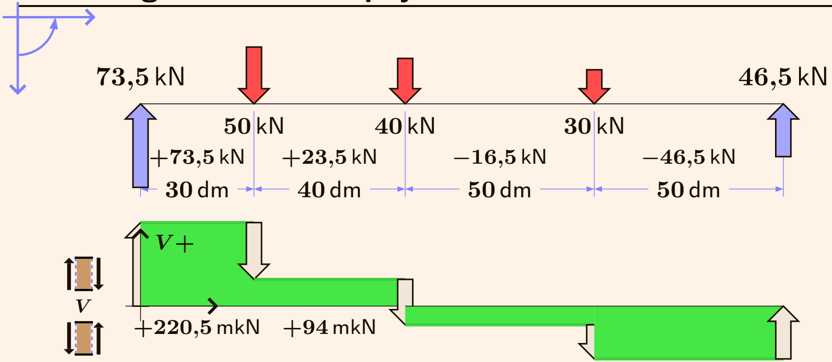
$$M(7\,\mathrm{m}) - M(0\,\mathrm{m}) = \int_0^7 V\,\mathrm{d}x$$

Tres cargas entre dos apoyos



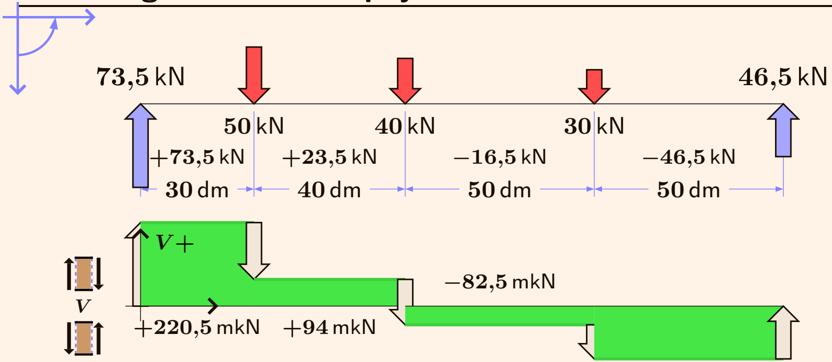
$$\text{area} = \mathbf{73{,}5}\,\text{kN} \times \mathbf{3}\,\text{m} = \mathbf{220{,}5}\,\text{mkN}$$

Tres cargas entre dos apoyos



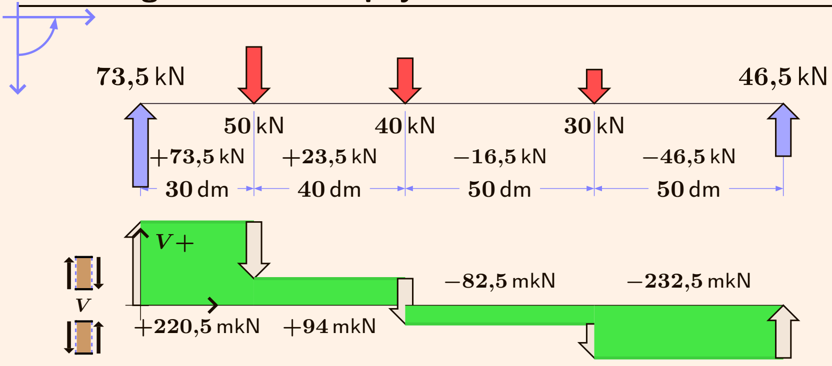
$$\text{area} = 23{,}5\,\text{kN} \times 4\,\text{m} = 94\,\text{mkN}$$

Tres cargas entre dos apoyos



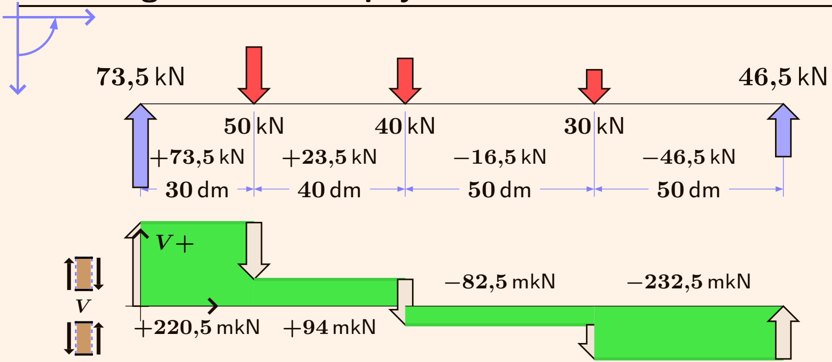
$$\text{area} = -16{,}5\,\text{kN} \times 5\,\text{m} = -82{,}5\,\text{mkN}$$

Tres cargas entre dos apoyos



$$\text{area} = -46{,}5\,\text{kN} \times 5\,\text{m} = -232{,}5\,\text{mkN}$$

Tres cargas entre dos apoyos

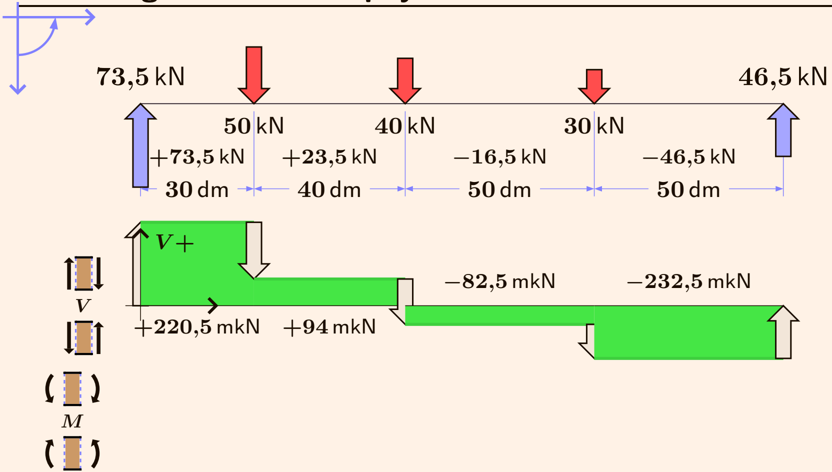


Comprobación:

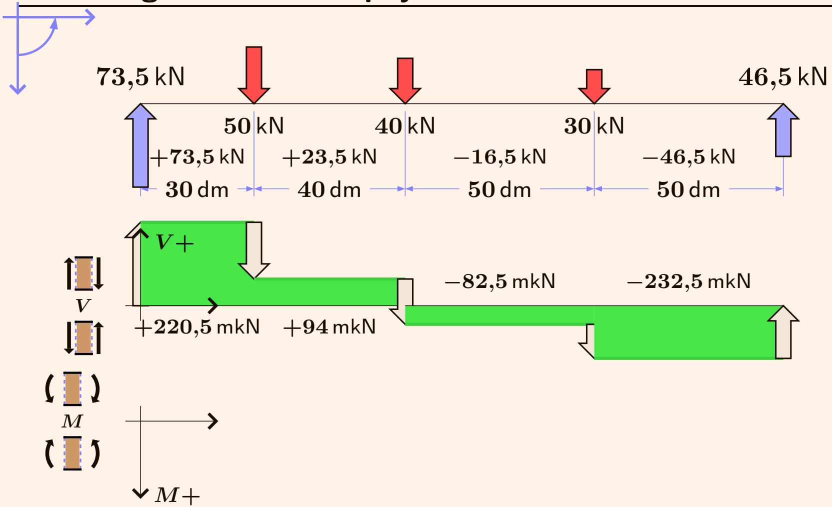
area total $= 220{,}5\,\text{mkN} + 94\,\text{mkN} - 82{,}5\,\text{mkN} - 232{,}5\,\text{mkN} = -0{,}5\,\text{mkN}$

(¡Debería dar cero! La precisión es aceptable. . .)

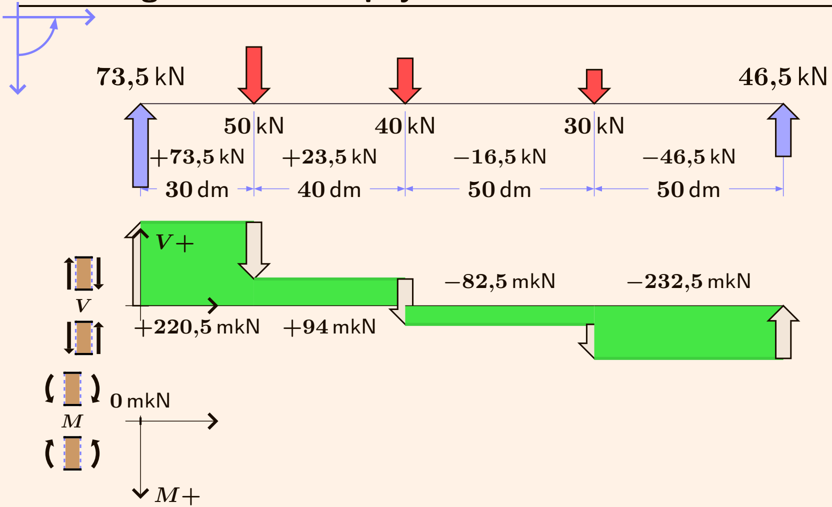
Tres cargas entre dos apoyos



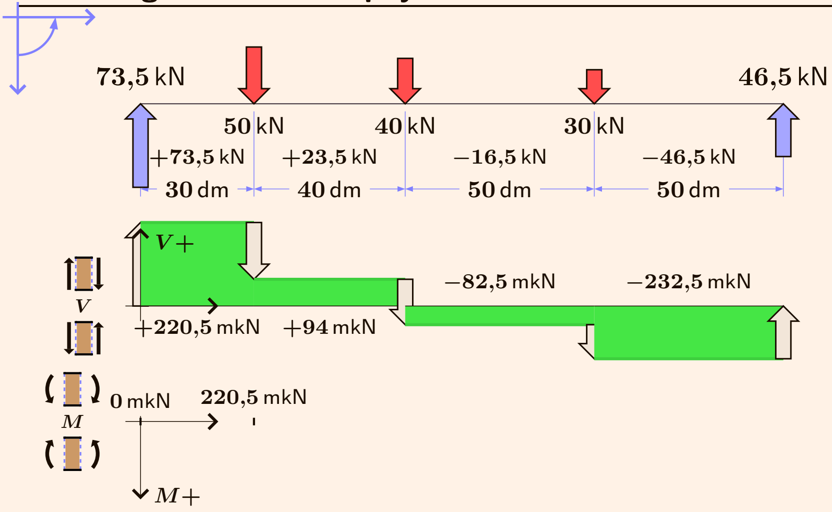
Tres cargas entre dos apoyos



Tres cargas entre dos apoyos

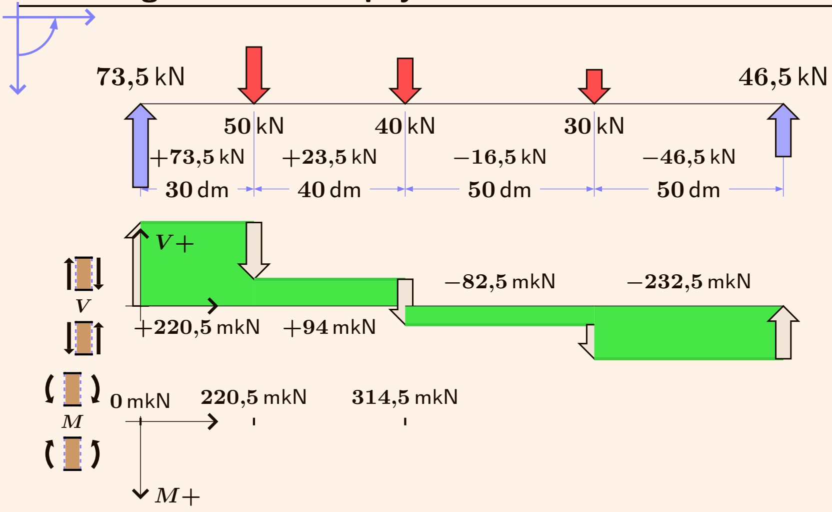


Tres cargas entre dos apoyos



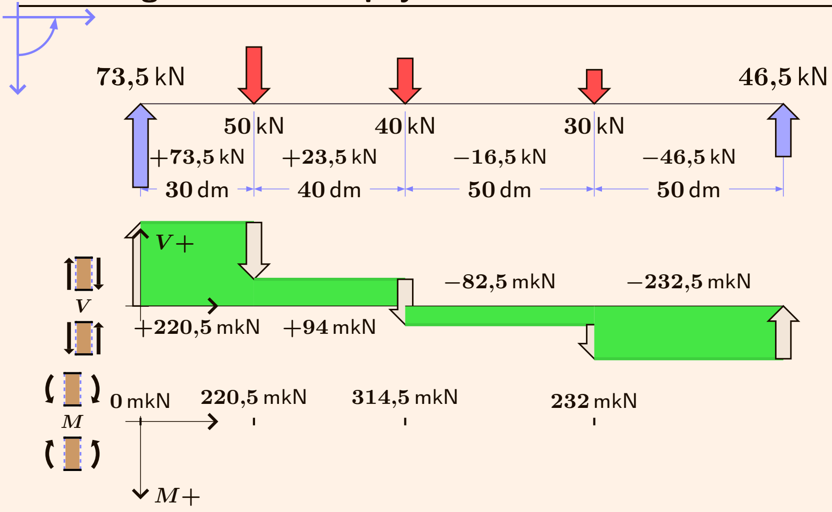
$$M(3\,\mathrm{m}) = M(0) + \int_0^3 V\,\mathrm{d}x \;= 220{,}5\,\mathrm{kN}$$

Tres cargas entre dos apoyos



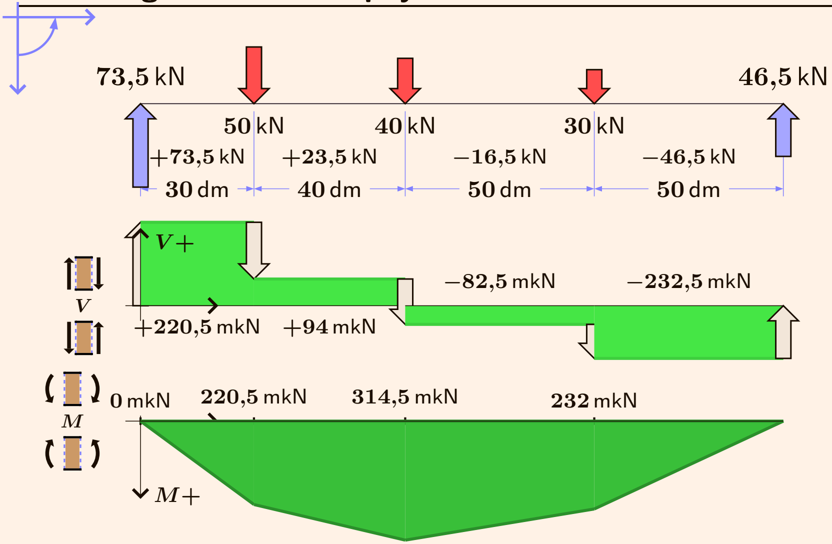
$$M(7\,\mathrm{m}) = M(3) + \int_3^7 V\,\mathrm{d}x = 220{,}5\,\mathrm{mkN} + 94\,\mathrm{mkN}$$

Tres cargas entre dos apoyos



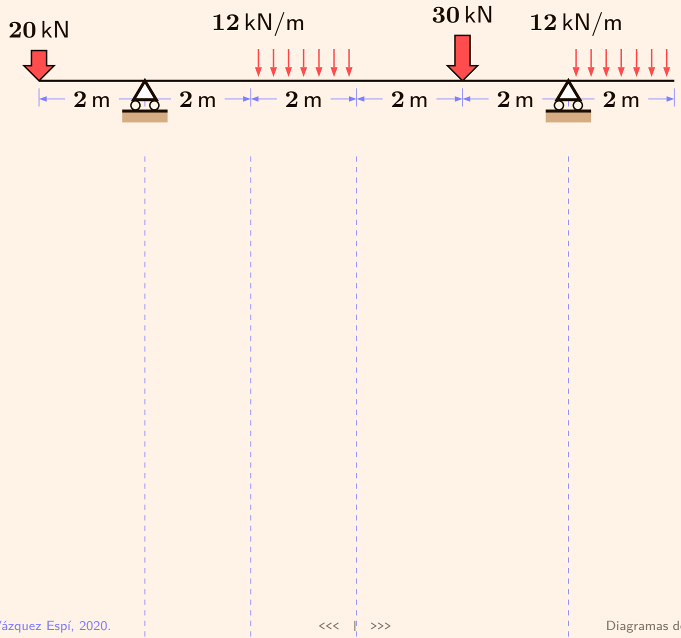
$$M(12\,\mathrm{m}) = M(7) + \int_7^{12} V\,\mathrm{d}x = 314{,}5\,\mathrm{mkN} - 82{,}5\,\mathrm{mkN}$$

Tres cargas entre dos apoyos

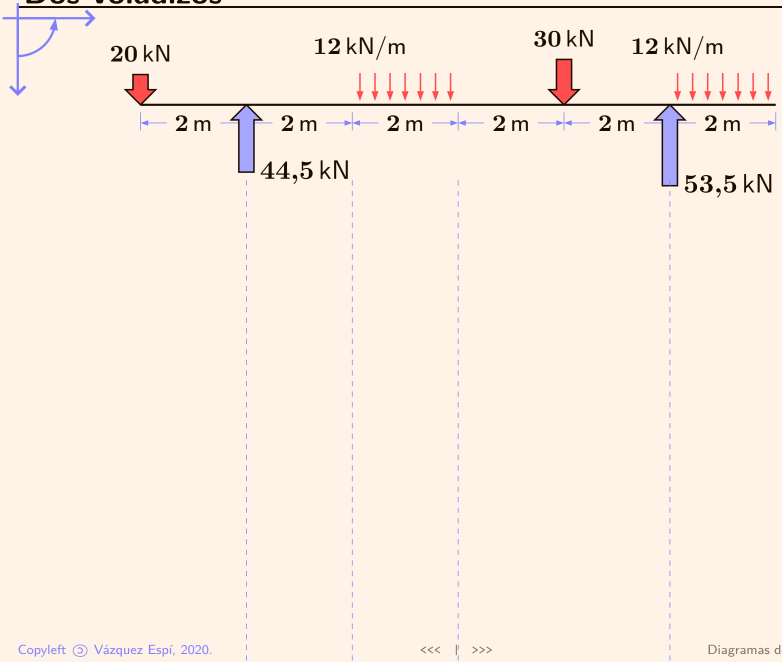


$$M(12\,\mathrm{m}) = M(7) + \int_7^{12} V\,\mathrm{d}x = 314{,}5\,\mathrm{mkN} - 82{,}5\,\mathrm{mkN}$$

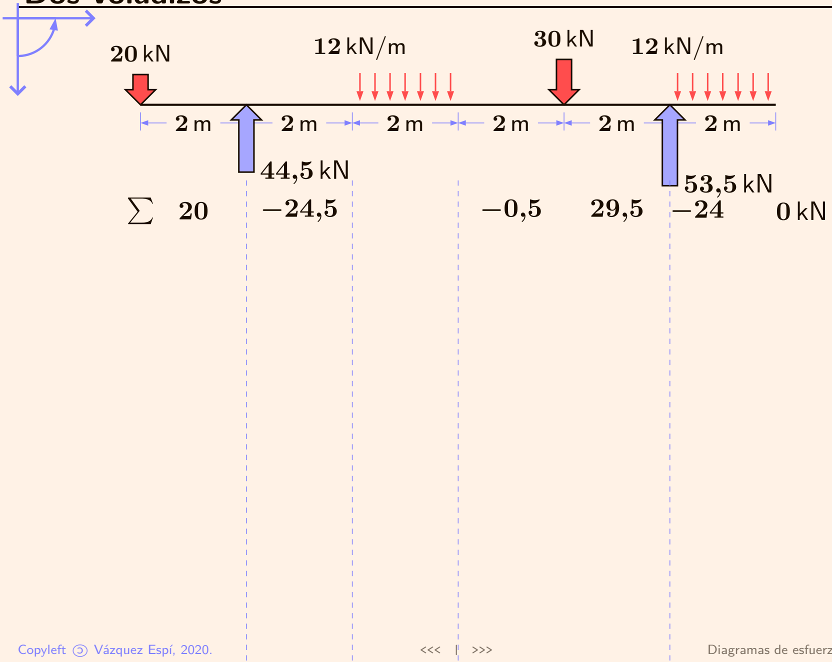
Dos voladizos



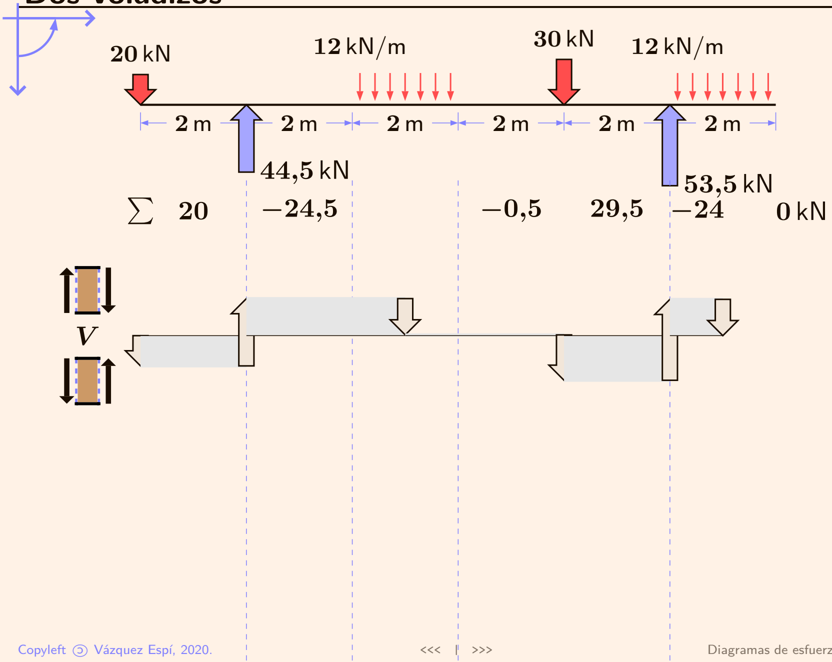
Dos voladizos



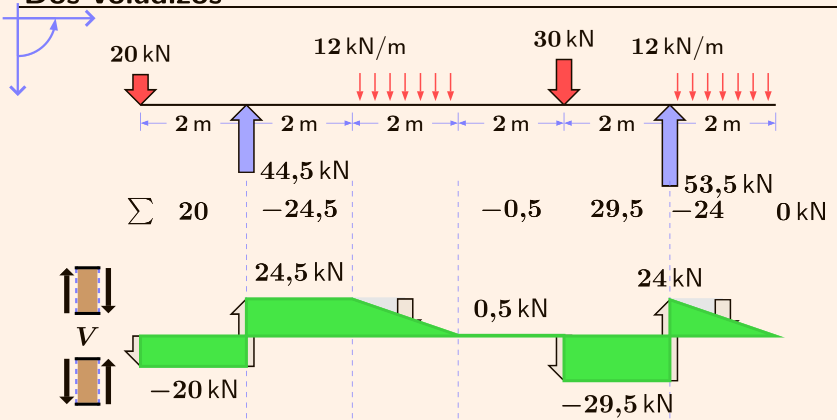
Dos voladizos



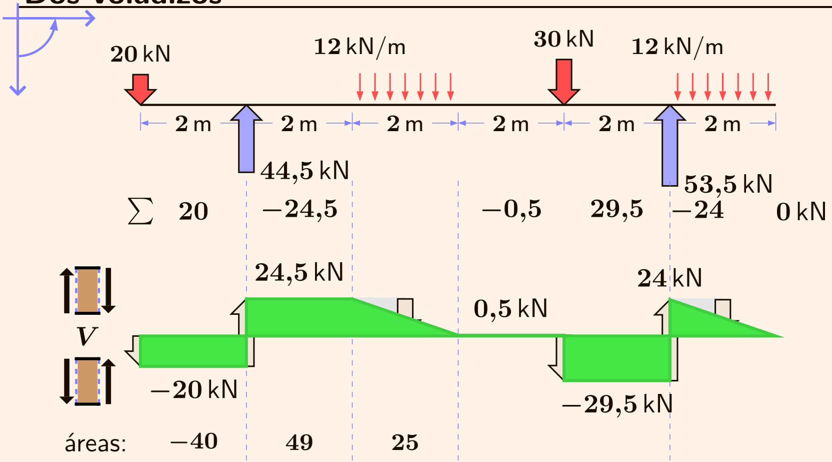
Dos voladizos



Dos voladizos



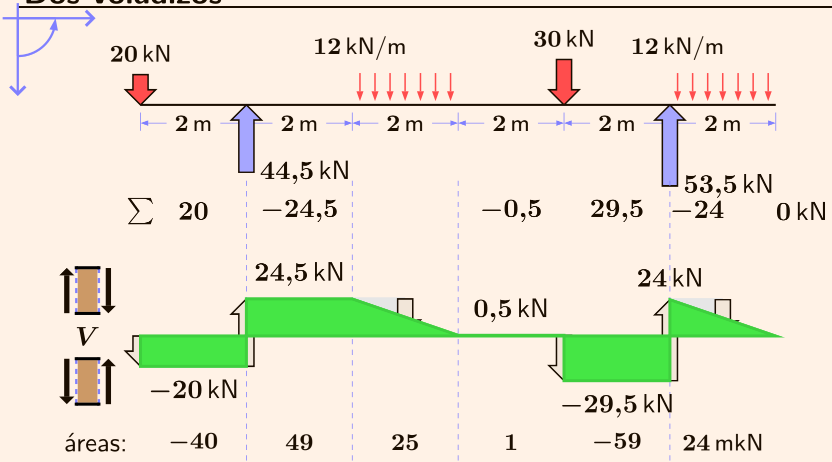
Dos voladizos



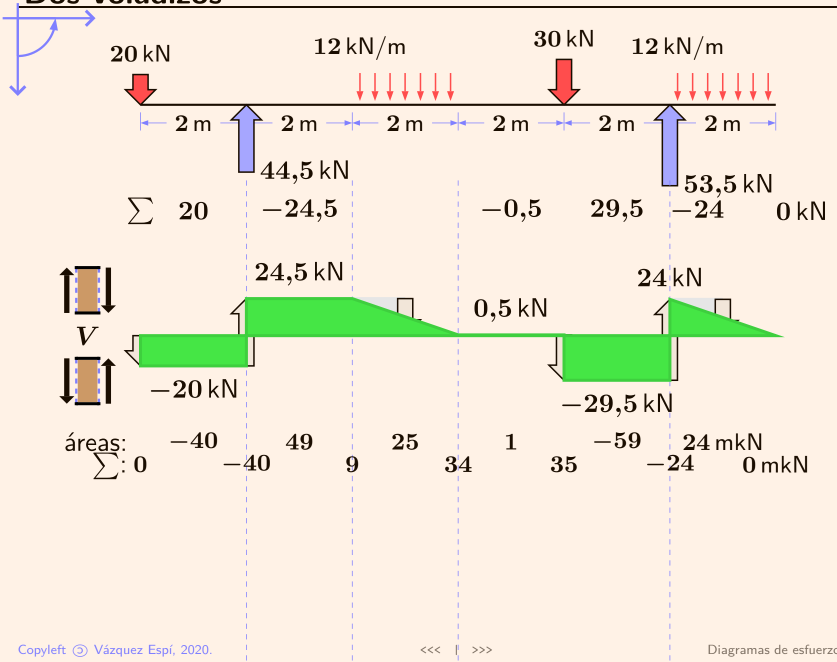
Trapecio:

$$\frac{24{,}5 + 0{,}5}{2}\,\mathrm{kN} \times 2\,\mathrm{m} = 25\,\mathrm{mkN}$$

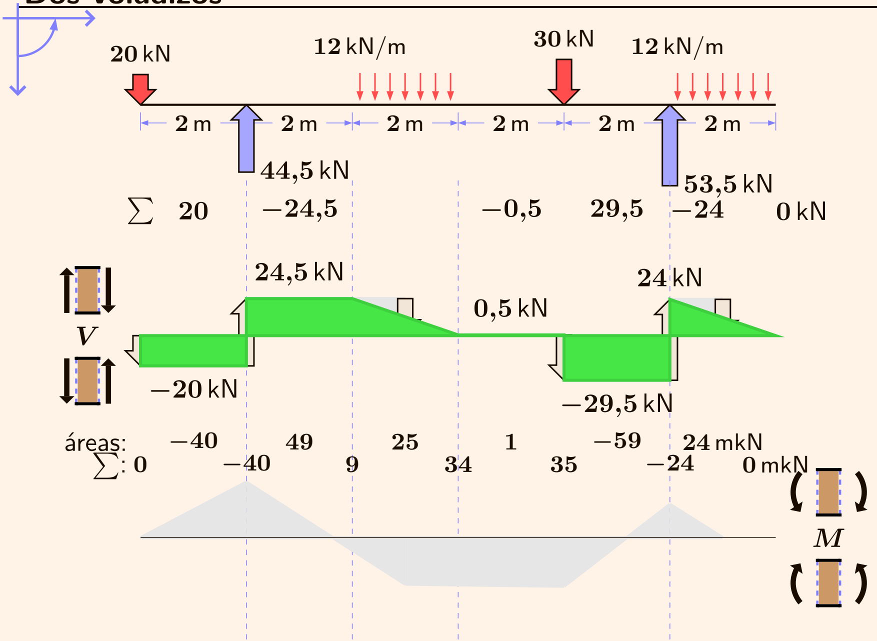
Dos voladizos



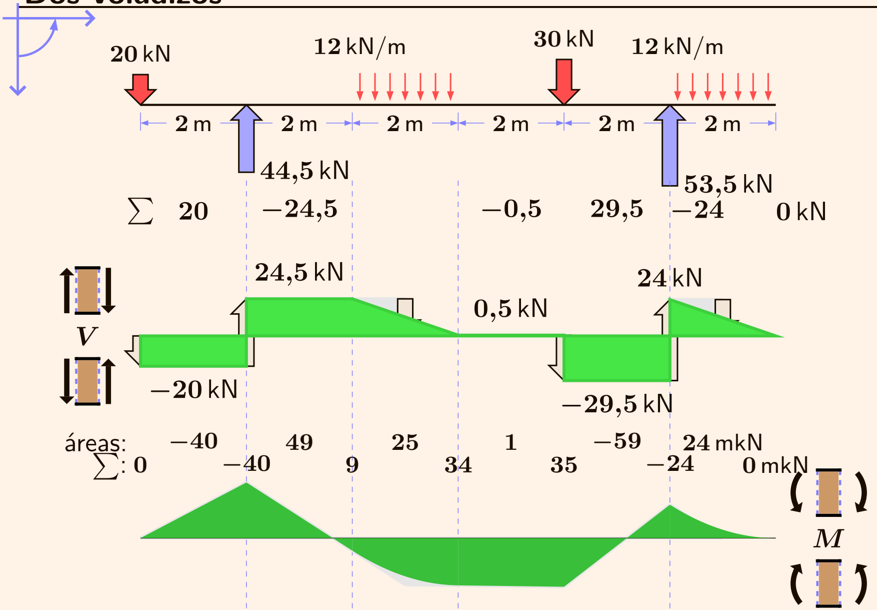
Dos voladizos



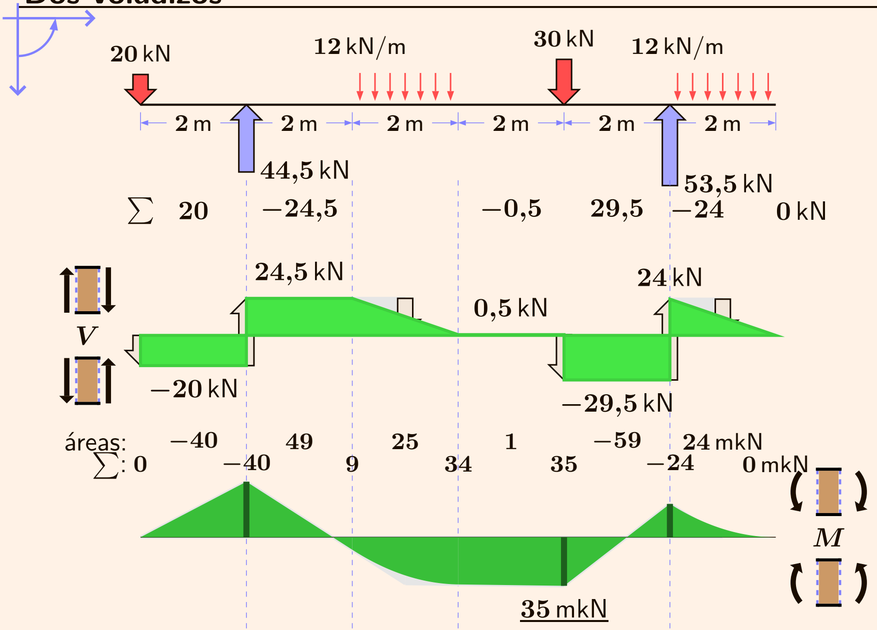
Dos voladizos



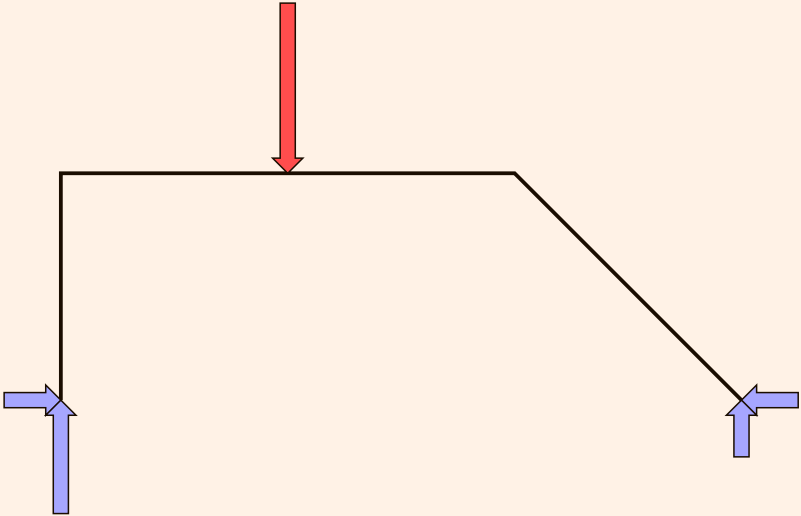
Dos voladizos



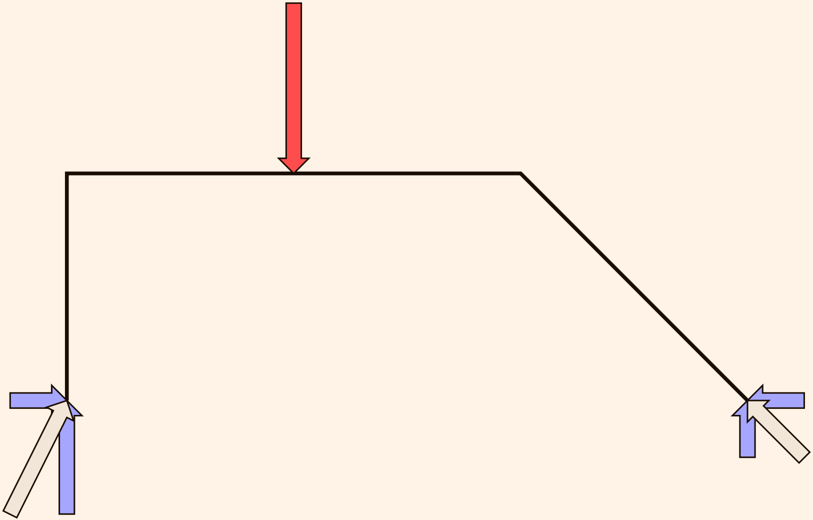
Dos voladizos



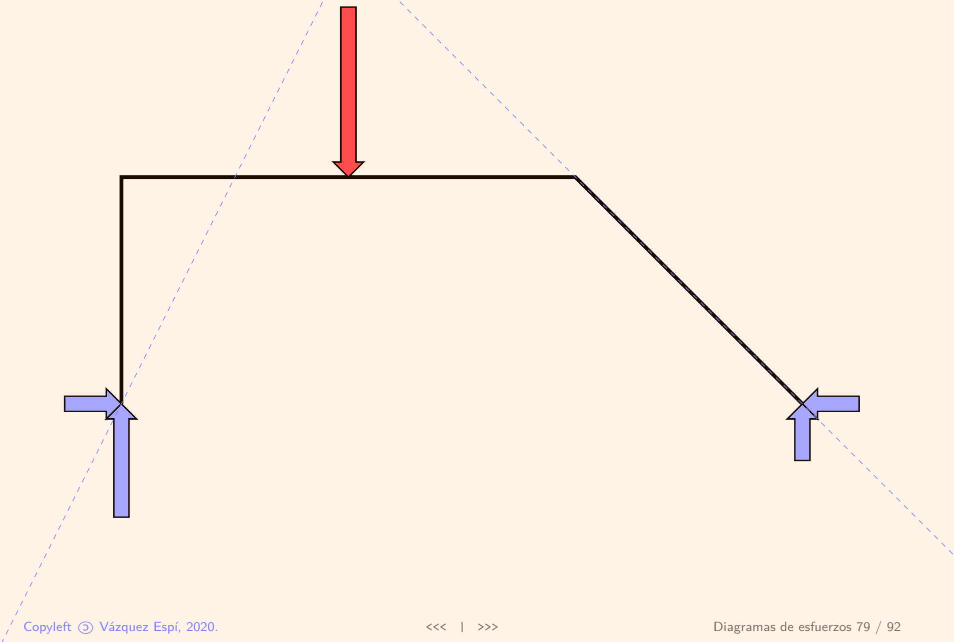
Directrizes poligonales



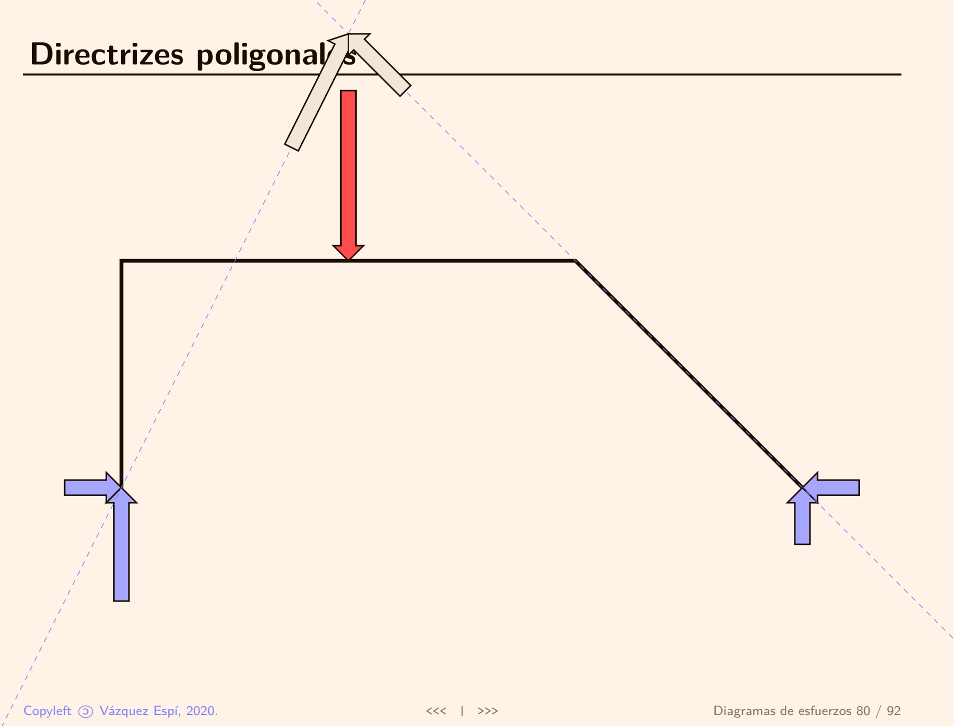
Directrizes poligonales



Directrizes poligonales



Directrizes poligonales



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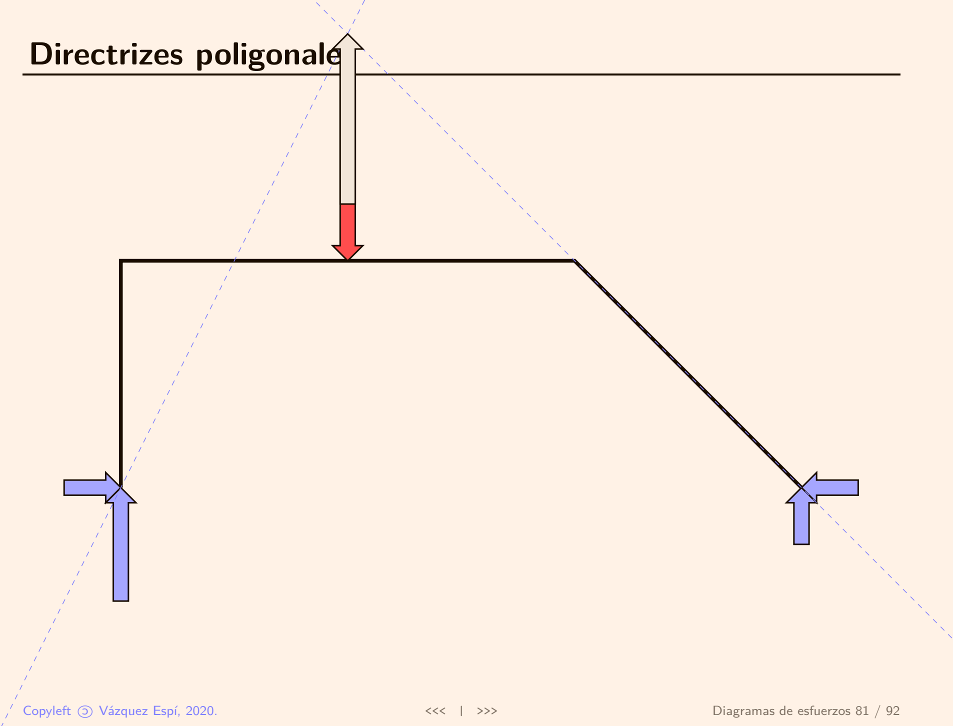
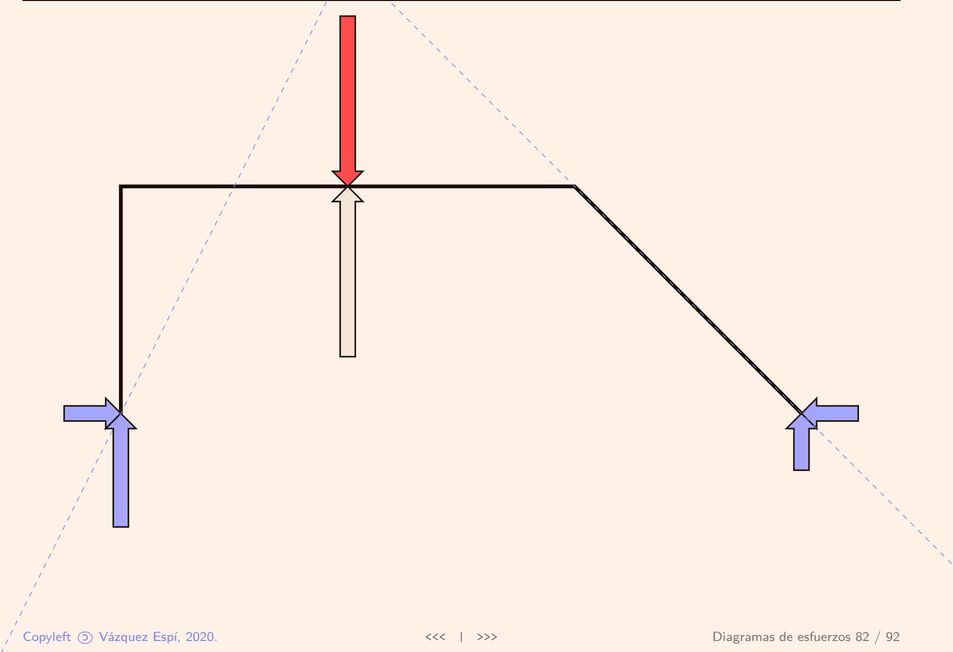
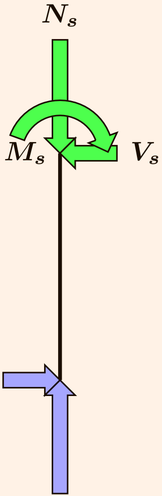


Directrizes poligonales



Directrizes poligonales



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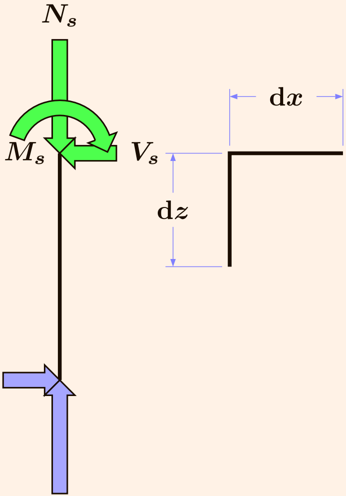
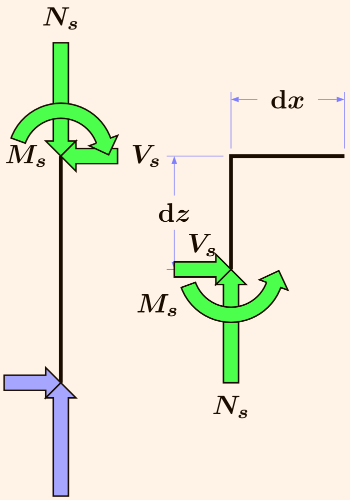
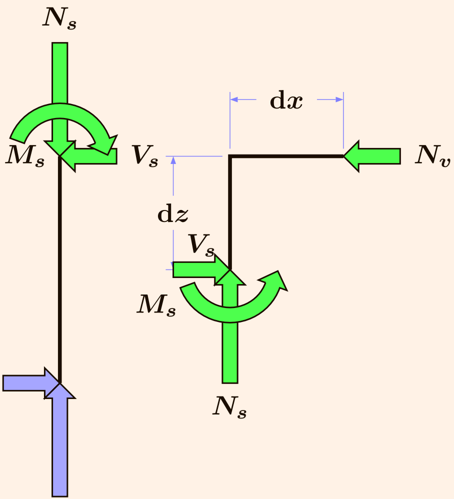


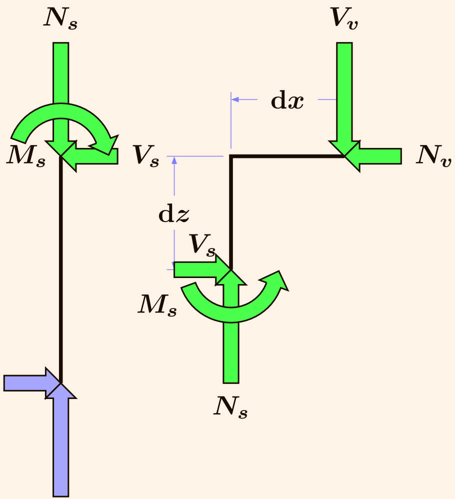
Directrizes poligonales



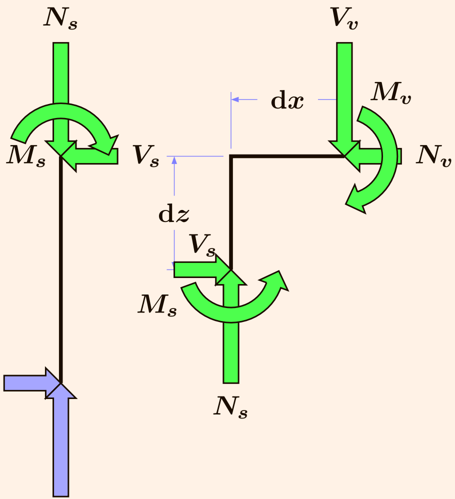
Directrizes poligonales



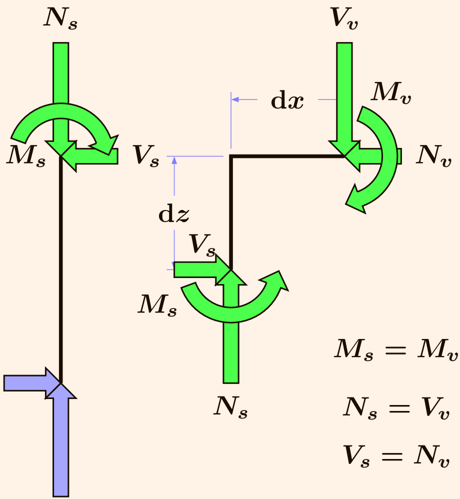
Directrizes poligonales



Directrizes poligonales



Directrizes poligonales

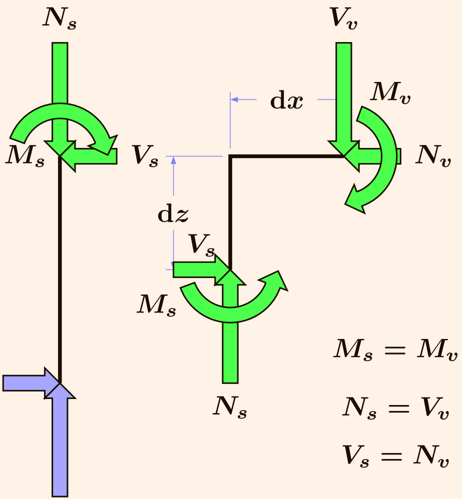


$$M_s = M_v$$

$$N_s = V_v$$

$$V_s = N_v$$

Directrizes poligonales



$$M_s = M_v$$

$$N_s = V_v$$

$$V_s = N_v$$

En general:

$$M_s = M_v$$

$$\vec{N}_s + \vec{V}_s = \vec{N}_v + \vec{V}_v$$

Otros ejemplos

- Un ejemplo de viga “quebrada” (como una escalera) con varias definiciones de las acciones (permanentes, variables, etc):

  http://habitat.aq.upm.es/gi/mve/mmcyte/h-ejemplo-d1.pdf

- Una amplia variedad de ejemplos sencillos (y bastante abstractos):

  http://www.aq.upm.es/Departamentos/Estructuras/e96-290/doc/p-diag-0708b.pdf

Diagramas de esfuerzos
(Funiculares como diagramas)

Mariano Vázquez Espí

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