A historical review of truss-like structures optimization theories since Maxwell until now

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ABSTRACT

The optimum use of materials became a key objective of engineering after Clausius warned about the exhaustion of mineral resources in 1885. Structural optimization stems from the approach of Maxwell (1870) and evolved thanks to the contributions of Michell, Cox and many other authors until 1965. Thereafter, new approaches, postulated by Hemp, Prager and Rozvany among others, have focused exclusively on the structure above the foundation. Within this framework, the present article analyses in-depth the competing approaches revealing different, practical results according to each theory. In order to illustrate the main conclusions, a number of historical examples are included. The profound significance of the conceptual scheme of the original Maxwell-Michell approach to reaching a sound structural design theory is highlighted: a scheme that computes all the physical costs incurred in each design, including foundation costs.

1 INTRODUCTION

The aim of the present article is to review the existing structural optimization theories with the purpose of looking for a common basis to formulate problems of structural design with actual practice in mind. The problems considered are restricted to the well known problem of design truss-like structures, i.e., those composed with members of constant cross-section and joints. In particular, the problem of designing the topology of a continuous medium, known as topology optimization, is not considered, given its very different nature, even though good solutions for this problem bear a visual resemblance to the optimal solutions for the former. The main reasons that support the need for a critical review of this topic are: a) the prevalence of some confusion in the literature about the main objectives of structural design and of structural optimization; b) the misunderstanding of Michell's seminal article of 1904 [1], in spite of the fact that almost all authors acknowledge the milestone character of this work; c) the existence of several approaches or theories about structural optimal shapes, e.g., the Maxwell-Michell theory and the Prager-Hemp-Rozvany theories, from which different outcomes can be obtained for equivalent problems, creating all sorts of controversy and confusion—more than half a century ago, Cox [2] anticipated the possibility of this controversy which has led to several public disagreements in the recent past (see, e.g., [3])—; d) and, as a matter of fact, 'structural engineering practice falls short behind of adopting optimization based design procedures' [4].

In order to introduce the framework as clear as possible, this work starts by attempting to fix context, concepts and theoretical basis and defining a common nomenclature. Section 2 is devoted to specifying what the day-to-day working goal of a structural designer is, and what kind of results she or he expects from tools as optimization methods, putting it in the broader perspective of Clausius [5]. Next, Section 3 is dedicated to introduce the mathematical basis of Maxwell's and Michell's structural design theories in contemporary terms. In section 4, a brief history of the evolution of theories after the work of Maxwell [6] is recalled, putting it in the broader perspective of the cost and efficiency concepts, and remainingin Maxwell's own realm, avoiding reinterpreting his ideas with our modern interpretations of terms or concepts. In section 5, the basic formulations of competing approaches and the differences are described. In section 6, several specific problems and solutions are examined to underline that these differences are relevant. Finally, conclusions and suggestions are exposed.

2 THE STRUCTURAL DESIGN CLASS OF PROBLEMS

As in any thermodynamic system, the definition of *the structure to be analysed* is arbitrary to some extent. The analyser can be interested at some moment in the performance of a substructure and she or he can re-arrange the former partition as appropriate. The problem pointed out by Prager and Rozvany [7] illustrated in Fig. 2(a) of p. 12, constitutes a good instance of the latter case as the analyser focuses on the bars that transmit a load to a vertical wall, delaying the analysis of the wall itself. In fact, this is customary and very useful in structural analysis but, *is it reasonable in structural optimization?* As this work will prove, *it is not because of the physical cost.*

There is no uncertainty about what composes the structure from a thermodynamic point of view regarding physical cost: everything that has a physical cost in order to build and to use it.

At the very beginning of the design process, only the *useful* actions —the design purpose—, the place and the requirements of any kind that the structure has to fulfill are given to the designer as the *design problem* data. And, as pointed out by Hemp [8, p.1], 'the theory [of structures] ought to be in a position to tackle the design problem directly, that is, to begin with the given forces and to produce by calculation the best structure that will safely carry them. [. . .] this [is a] little developed branch of the theory [in 1958, but] which may lead to developments of knowledge, such as to make direct structural design a practical art of the normal techniques of engineering'.

Generally, there are multiple feasible solutions to the same design problem, although usually, the purpose is to obtain the best possible solution, or at least, a *better* one with each new attempt. Moreover, as each solution must fulfill both safety and stiffness requirements, the term of comparison must be another indicator. Herein, following Michell, *physical cost* is adopted, preferring the solution that requires a lower cost. Nonetheless, of course, there are other possible figures of merit (see, e.g., [9] on "morphological indicator").

As a first approach, the physical cost can be represented by the self-weight of the structure, as many costs during the manufacturing process are approximately proportional to it: CO_2 emissions, mineral resources consumption, etc[4]. Additionally, if the materials used allow very large scopes with regard to the size of the structural problem considered, the self-weight of the different solutions of the problem is negligible with respect to the useful loads (all the rest), and as a consequence, the self-weight can be disregarded at first instance *as a load* [see 10].

This realm is where Maxwell's or Michell's ideas should be placed; they are not concerned about the self-weight as a load, but very interested in the other costs relative —or proportional— to the self-weight.

3 THE MAXWELL-MICHELL THEORY ON STRUCTURAL DESIGN

In 1870, Maxwell [6, pp. 175–177] shows that in a system of points in equilibrium under the action of 'attractions' and 'repulsions', the sum of the products of each attraction by the length on which it acts is equal to the same sum applied to the repulsions. Then, assimilating a frame to a set of internal forces overlaid by a set of loads and reactions, indirectly he describes an invariant for all the solutions to a structural problem so defined (i.e., a fully defined set of external forces in equilibrium: a *Maxwell problem*, (see definition 1 in Table 1)): the virtual work of the external forces when the space undergoes a uniform contraction that reduces it to a point, named hereafter as *Maxwell number* $\mathcal{M}$.

Later, in 1904, Michell [1] returns to Maxwell's work, and shows explicitly the above invariant. More interestingly, he demonstrates that the volume of a *full-stressed* structure solving a Maxwell problem is minimum if the sum $\mathcal{V}$ of the products of the absolute value of the internal force, e, of each bar by its length, ℓ, is minimum, i.e., if $\mathcal{V} = \sum_i |e_i|\ell_i$ is minimum. Michell named this magnitude ($\mathcal{V}$) as 'Quantity' and it has proven to be essential in the theory of structural design [see 11]. In this work it is named 'stress volume' (see definition 5 in Table 1), denomination that arises from the fact that $|e|\ell = |\sigma|A\ell$, being A the cross-sectional area of a typical truss bar and σ its stress, hence $\mathcal{V}$ can be viewed as the stress volume of the framework. Other denominations have been used: 'structural work' [12], 'quantity of structure' [13], 'absolute pertinacity' [14], 'internal load transportation measure' [15], or 'load path' [16].

Furthermore, the components of the stress volume can be separated in the tension $\mathcal{V}^+$ and the compression $\mathcal{V}^-$ contributions, so that $\mathcal{V} = \mathcal{V}^+ + \mathcal{V}^-$, which constitute a very useful expression for some calculations.

Maxwell's and Michell's theories are outlined in Table 1, in which names and definitions are proposed by the Authors of this work. Additionally, a more detailed description can be found in [19]. Maxwell's findings are summarized from Definition 1 to Lemma 4 in the table. Michell's findings come from Lemma 8 and his two fundamental theorems, 9 and 10. From Michell's Lemma 8, the geometrical volume of a fully-stressed truss is $V = \mathcal{V}^+/f_+ + \mathcal{V}^-/f_-$, where f_+ and f_- are the allowable stress of tension and compression respectively. In fact $1/f$ is the geometrical volume per unit of stress volume, i.e., $1/f_+$, $1/f_-$ are unitary costs. A similar definition can be given for structural weight (being the unitary costs ρ_+/f_+, ρ_-/f_- and denoting ρ the specific weight of materials used), embodied energy of materials, etc.

In his article, Michell develops the examples considering the geometric volume as the functional to be minimized. However, in the sequel the stress volume $\mathcal{V}$ will be used following Michell's lemma (lemma 8). The main reason for using said functional is that with Corollary 6 the search for the optimal truss can be stated in short as :

Table 1. The Maxwell-Michell theory, contemporary nomenclature and definitions

Definition 1. ***Maxwell problem:*** *To find a feasible structure for a given set of known, external forces in equilibrium. Each external force must be determined in position, direction and magnitude.* (Maxwell [6, pp. 173, 176–177])

Definition 2. ***Maxwell structure:*** *Any set of internal forces —defined by their magnitude 'e' (taking compression as negative) and their two application points, being 'ℓ' the distance between them—, such that added to the external forces of a structural problem to form a complete set of internal and external forces, this latter satisfies that every subset of all the forces acting at the same point has a nil resultant.* (Maxwell [6, p. 161])

Definition 3. ***Maxwell number $\mathcal{M}$ of a Maxwell structure:*** *The negative of the virtual work done by the internal forces when the space undergoes a homogeneous deformation that reduces it to a point,* $\mathcal{M} = \sum_i e_i \ell_i = \int e \, \mathrm{d}\ell$. (Maxwell [6, p. 177])

Lemma 4 (Maxwell's Lemma). *The Maxwell number of all the structures that solve a given Maxwell problem is constant.* (Maxwell [6, p. 177]; Michell [1, Eq. (1)]; Owen [17, p. 50: 'static constant'])

Definition 5. ***Stress volume $\mathcal{V}$ of a structure:*** $\mathcal{V} = \sum_i |e_i| \ell_i = \int |e| \mathrm{d}\ell$

Corollary 6. *The Maxwell number is the difference between stress volumes of tension and compression,* $\mathcal{M} = \mathcal{V}^+ - \mathcal{V}^-$. (Maxwell [6, p. 176]; Michell [1, Eq. (1)])

Corollary 7. *In any Maxwell problem, any feasible variation of a feasible solution is such that the variation of tension volume equals the compression one:* $\delta\mathcal{V}^+ = \delta\mathcal{V}^-$. (Parkes [18, p. 163])

Lemma 8 (Michell's Lemma). *For any cost* $\mathcal{C}_i$ *defined as* $\mathcal{C}_i = k_+\mathcal{V}^+ + k_-\mathcal{V}^-$ *with* $k_+ \geq 0$, $k_- \geq 0$ *and* $k_+ + k_- > 0$, $\delta\mathcal{C}_i = 0 \Leftrightarrow \delta\mathcal{V} = 0$ *holds for a Maxwell problem, i.e., the solution of minimal* $\mathcal{V}$ *will be of minimal* $\mathcal{C}_i$. (Michell [1, Eq. (3)]; Hemp [8, p. 4, Eq. 7].)

Michell's theorems
Let Δ be a finite, strictly positive strain. Let $\mathcal{D}$ be the set of bounded, continuous displacement fields d such that the strain ε^d of the field d at all points and directions of the considered domain, Ω, fulfils $|\varepsilon^d| \leq \Delta_d$. Let $\mathcal{S}$ be the set of all Maxwell structures for a Maxwell problem enclosed into Ω.

Theorem 9 (Michell's first theorem).

$$\forall (d, A) \in \mathcal{D} \times \mathcal{S} : \quad \frac{W^d}{\Delta_d} \leq \mathcal{V}(A) \qquad (1)$$

where W^d *is the virtual work of the given external forces of the Maxwell problem when the domain* Ω *undergoes the displacement field* d. [1, p. 590].

Theorem 10 (Michell's second theorem). *If a pair* $(T, M) \in \mathcal{D} \times \mathcal{S}$ *exists for* Ω, *such that* $\varepsilon_i^T e_i^M = \Delta_T |e_i^M|$ *for every member i then 'the truss M attains the limit of economy of material' in* Ω, $\mathcal{V}(M)$ *'is a minimum, and consequently from [Michell's lemma] the volume of material in the frame M is also a minimum.'* [1, p. 591].

Find $\min(\mathcal{V}^+ + \mathcal{V}^-)$ subject to $\mathcal{V}^+ - \mathcal{V}^- = \mathcal{M}$, for the given constant $\mathcal{M}$.

Any other cost $\mathcal{C}_i$ for unitary cost k_+ and k_- is defined for each solution as:

$$\mathcal{C}_i = k_+\mathcal{V}^+ + k_-\mathcal{V}^- \qquad (2)$$

and can be computed from $\mathcal{V}$ (of a Maxwell structure) and $\mathcal{M}$ (of the Maxwell problem) as [2, p. 87, Eq.(121)]:

$$\mathcal{C}_i = \frac{1}{2}\left\{(k_+ + k_-)\mathcal{V} \ + \ (k_+ - k_-)\mathcal{M}\right\} = \frac{k_+ + k_-}{2}\mathcal{V}\left(1 \ + \ \frac{k_+ - k_-}{k_+ + k_-}\frac{\mathcal{M}}{\mathcal{V}}\right) \tag{3}$$

The key point here is that an optimal Maxwell structure for $\mathcal{V}$ is *also* an optimum for any cost $\mathcal{C}_i$. That is, an optimal Maxwell structure for $\mathcal{V}$ will be of minimal volume or weight independently of the allowable stresses or the specific weights of materials; hence, its shape will be the same shape for any cost $\mathcal{C}_i$ [2, p. 116]

Moreover, Michell identifies a condition that makes the volume of a structure that solves a given Maxwell's problem to be an absolute minimum in the geometrical domain in which alternative structures can exist, which is summarized in Theorem 10. After that, he shows certain types of structures that fulfil the indicated condition, showing a geometric condition for non-trivial frames: '. . . those whose bars, both before and after the appropriate deformation, form curves of orthogonal systems'.

Is Michell's second theorem an optimality criterion? That is, is this theorem (number 10 in Table 1) a necessary and sufficient condition that must fulfil an optimum? In order to answer this question, the requirements to fulfil it are examined. Michell's optimal condition in Ω requires:

1. a finite boundary strictly positive Δ_T for the field T, and
2. $\varepsilon_i^T e_i^M = \Delta_T |e_i^M|$ fulfils by all members of the structure M.

Since Michell does not offer any proof of the existence of a pair (T, M) for every set of given external forces in equilibrium, he only considers his condition as a sufficient one [1, p. 589][20, p. 13,70ss]. In spite of a sustained research effort on this subject, it has not been proven that Michell's condition is also a necessary one [see 21, 22, 23]. Highlighting the fact that this argument refers only to Michell's original condition for Maxwell's problems.

Given a Maxwell's problem, finding the pair (T, M) is not generally a straightforward task. Firstly, a structure M has to be found. Secondly, proving that M is optimal requires finding a field T so the pair fulfills Michell's theorems. Thirdly, if a field T is not found, proving that the claimed M is not optimal requires finding a better structure.

In the last part of his article, through the application of the mentioned theorems and condition, Michell shows which are the optimal forms for some Maxwell problems. It is worth noting again that Michell's theory is built on Maxwell's theory, and thus, they have not defined supports or kinematic conditions, but only a balanced set of external forces including reactions. It is worth noting that this approach to the structural problem was customary at the time [see 24].

The resulting optimal structures are characterized by their orthogonality and by the existence of continua regions with an infinite number of elements. Layouts based on Michell's theorems and condition, or with some similarities with those layouts —infinite number of bars, orthogonality— become now a category, being known as 'Michell trusses', 'Michell nets', 'Michell frames', and so on.

Rozvany et al. [25] declare: '*classical Michell trusses* are not practical because they usually have an infinite number of members, ignore buckling, consider one load condition only, and are, as a rule, unstable mechanisms. However, they constitute a *classical field*, which has been investigated by many researchers. Michell trusses are also used regularly as *benchmarks* for checking on the validity of various numerical methods'.

4 HISTORICAL BACKGROUND

After its publication, Michell's work remains unnoticed for about half a century until J. Foulkes [26] tells H. L. Cox about the existence of this work [27, p. 156], who reads the article and realizes its theoretical importance [see 28, p. 2][2]. In the following years, other researchers like Hemp [8], Parkes [18] and Owen [17] begin to disclose and to arrange in a clear mathematical form the consequences of Michell's work. At the same time, a number of researchers formulate methods or present demonstrations that constitute milestones in relation to structural optimization.

A first milestone was the demonstration by Sved [29] that the minimum weight truss for problems with redundant, kinematic conditions is a statically determinate one. This theorem will be very useful in future research into this class of problems (to which Maxwell's problem does not belong). Sved, Australian as Michell, does not cite Michell's findings.

A second milestone was the formulation of an optimality criterion for trusses of minimum weight by Prager [30, 31], who introduces kinematic conditions in the problem so a necessary and sufficient condition for optima can be attained [32].

A third milestone was the formulation of the ground structure method by Dorn, Gomory, and Greenberg [33], which influenced many studies afterwards. Although this method was formulated ignoring the work of Michell, the method shows how to find an optimal truss for problems restricted to a finite number of bars. Generally, this optimal truss cannot be an optimum in Michell's sense, but the work necessary to find restricted optima was affordable for said times. Furthermore, this method uses the standard structural analysis, with kinematic conditions, so its implementation is a simple extension of existing codes.

In 1964, Frei Otto founded the Institute for Lightweight Structures at the University of Stuttgart. His work and that of colleagues [15] on the "lightweight principle", open a new branch in this subject, resumed in three main concepts, named `TRA`, `BIC` and `LAN`. The connection with Michell's approach is the first one: `TRA` is a measure of 'load transportation', distinguishing among several types: e.g. external, internal and relative. The first type is defined as the summation of the products of the absolute value of the loads by its distance to the nearest support: $\mathtt{TRA}_a = \sum |Q||s|$; the second one is directly the *stress volume*: $\mathtt{TRA}_i = \sum |e|\,\ell = \mathcal{V}$, being the relative `TRA` the quotient $\mathtt{TRA}_r = \mathtt{TRA}_i/\mathtt{TRA}_a = \mathcal{V} \div \sum |Q||s|$. Actually, Otto and coworkers, without a direct knowledge of Michell's work—in [15], 'Michell' is 'Mitchell' and his work is dated in 1908—, also arrive to Michell's 'quantity'. However, Maxwell's lemma is ignored, so Otto's formulation of structural design problem is not ever the same than that of Michell's. The book by Otto and co-workers was published in 1998, but as Otto says in the chapter 'Remarks on how this work came about', his work began much earlier, in 1946. Albeit little known, it can be considered as a very inspiring manual about structural design ([34]).

In 1973, Hemp [20] presents an optimality criterion for minimal volume trusses, depending on the allowable stresses of the materials in tension and compression, and subject to the kinematic conditions imposed by the supports of the structure. A decade after that, in 1984, Rozvany [35] presents a derivation of the Prager-Shield theory of optimal plastic design [36] using the notion of 'structural universe', obtaining a relaxation of the conditions of the Michell strain field test in some specific regions. Later, he introduces his 'optimality criteria', [e.g. 37], which confirms those of Hemp. These optimality criteria, which conversely to that of Michell depend on the kinematic conditions of the problem, become the current point of view of most researchers, relegating the Michell sufficient condition to a particular case of this new approach (or directly indicating an error in Michell's derivation—'Rozvany demonstrated that there was an error in Michell's derivation of optimality criteria' [38]—). However, actually, there are no shortcomings in Michell's theory, as discussed below.

In [19], extensions to include friction forces and dumped mass were suggested and managed. In [10, Ch. 5] an extension of Michell's theory to tackle with the self-weight as load considering catenary bars with constant cross-section was proposed (first published in [39]). A similar extension but with constant stress —and variable cross-section— was independently proposed in [40]. Recently the latter was used to research into optimal forms for bridges with self-weight load, several types of foundations, and dumped mass [41], for both fixed boundary and free load classes of problems, including friction forces.

According to Google Scholar, Michell's article has been cited 1768 times, which gives a rough idea of the amount of research invested on the subject. Despite this, the analytical solution for the optimal forms have only been found for a small set of simple problems. An extensive account of this effort can be found in the book by Lewiński, Sokół, and Graczykowski [42].

As shown throughout this review, finding the optimal solution for well-posed structural design problems is not a straightforward task. For that reason and with the day-to-day work in mind, some researchers have adopted the identification of the range in which very good solutions would be found as their main target, abandoning the search for optima [see, v.g., 19, 43, 44, 45]. Although these approaches are not the objective of this review, they are mentioned herein because of the advance they represent in practical terms.

5 THEORIES FOR PROBLEMS WITH KINEMATIC CONDITIONS

Prager [30] reformulates the problem of Michell [1] looking for minimal weight as follows:

> *Consider a truss structure able to carry the required set of forces with stresses in the tension members equal to f_+ and stresses in the compression members equal to $-f_-$. The weight per unit volume of the material used for the tension members is ρ_+ and of the material in the compression members is ρ_-. If, when the tension and compress members are subjected to virtual* strains $\varepsilon_+ = (f \cdot \varepsilon/\rho)(f_+/\rho_+)$ *and* $\varepsilon_- = (f \cdot \varepsilon/\rho)(f_-/\rho_-)$, *respectively* [note that $f = \frac{1}{2}(f_+ + f_-)$], *the resulting displacements satisfy the* kinematic conditions *imposed on the structure by its supports and any constraints that may exist on the permissible movement of supported loads, and no direct strain in the space within the structure lies outside these* limits, *then the structure has least weight of all competing structures. [32][46, p. 1783]*

Hemp [20] reformulates the problem looking for minimum volume, not for minimum weight, as follows:

> *A pin-joined framework has the least volume of material, if it can carry its given forces, with stresses in its tension members equal to f_+ and stresses in its compression members equal to $-f_-$ and if a virtual deformation of a region of space, in which the competing frameworks must lie, satisfies* the kinematic conditions imposed *on the framework[s] and gives strains of $f \cdot \varepsilon/f_+$ in its tension members, strains of $-f \cdot \varepsilon/f_-$ in its compression members and has no direct strain lying outside these limits.*

The subtle differences with Michell's theorems are that in this theory the problems present *kinematic conditions* and ,as a consequence, there are *two different absolute values bounding the virtual deformation*. Despite that Hemp says that the above description is the 'Michell's sufficient condition', in fact, it is not. This fact is surprising when Hemp's work of 1958 [8] is compared with the latter. In [37], Rozvany offers a personal, detailed description of the evolution of Hemp's ideas about this subject.

In both formulations there is no equilibrium condition over the given forces, so there are no Maxwell problems (definition 1) and Maxwell's Lemma does not hold (lemma 4). Furthermore, the cost of the supports is not considered and, consequently, the optimal solutions obtained from these three different criteria for equivalent problems are different. In fact, the optimal shape for a structure with the same kinematic conditions will be different for each cost definition ($\mathcal{C}_i$) [46, 47]; which constitutes a crucial difference with Michell's results.

Therefore, within this framework, in order to take into account the basis of all the theories and following Cox [2], two different types of structural problems can be defined depending on their target.

On the one hand, the 'free loading' problems, that corresponds to the named Maxwell problems, i.e. the reactions are known or designed—assuring external equilibrium— and are independent of the structure layout. Designing reactions is a normal practice in limit analysis, although that is done by means of the prescription of some internal forces in beams (see, v.g., Heyman [48, 'reactant line']).

On the other hand, the 'fixed boundary' problems, which present imposed kinematic conditions, and thus, in general, with unknown reactions that must be determined by structural analysis as usual, with values depending on the layout.

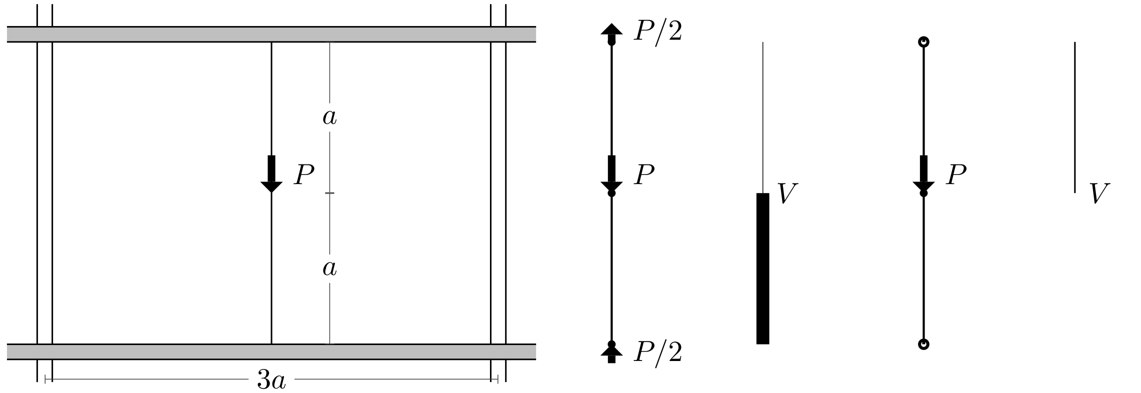
It is worth mentioning that both classes are not completely separated sets, as they share a common subset of problems: those which within the fixed boundary approach present statically determinate kinematic conditions. Nonetheless, Michell's theory uniquely applies to the free loading approach, and the shortcomings presumed by others authors are derived from an incorrect attempt to apply it to problems depending on kinematic conditions.

The researchers that rediscover Michell's theory have already warned of the different nature of these two approaches:

> *This result suggests that in some design problems a lighter structure might be obtained by adding loads to make the best 'minimum' structure possible, rather than by using a Michell structure. It must be remembered, nevertheless, that the reactions such as those at [fixed supports], are in any case carried by some other bodies acting as structures and the true picture of the economy achieved should include the abutments* (i.e. their cost). *(Owen [17, p. 64])*

> *When the supports are actually fixed, the nature of the design problem* is vitally altered. *The direction of the reactions at the supports are then determined in part by the structure itself, so that $\sum \bar{F}_i \bar{r}_i$ is variable [herein $\mathcal{M}$], and Clerk Maxwell's lemma, where still true, is of no use. (Cox [2, pp. 95–96])*

The fact that constitutes the main drawback of the 'fixed boundary' class is that the support cost is *different for each solution*, and thus, the comparison between the costs of the solutions is not possible when solely taking into consideration the cost of the structure it-self and ignoring the cost of the imposed kinematic conditions. Rozvany and Sokół [49] extends the optimal layout theory of



(a) The design problem (b) Free load version (c) Fixed boundary version

Figure 1. Comparison of the three approaches.

Prager and Rozvany [7] trying to fill this shortcoming, although it seems that the simple case of a foundation with friction forces is not covered by this extension [50].

The drawback with Michell's approach is that for representing all the cases of Hemp or Prager's approaches an infinite set of Maxwell's problems have to be considered for each set of definite reactions.

A very simple example will show the differences between the three approaches. Consider the problem of suspending a *new* weight P in the middle of a floor in a multi-storey building, see FIGURE 1.

The up and down beams have been designed for a load Q and so they were built. For a span of 6 m, Q will be of the order of 360kN. The new load P is $\frac{1}{6}$ of Q and will diminish the safety of the beams. The upper cable is made of steel and the bottom strut is made of wood. The proportion $f_+ \div f_-$ will be approx 17, and the proportion $\rho_+ \div \rho_-$, 10. Note that the feasible solutions are not statically determined.

In Michell's realm, the reactions will be fixed to $P/2$, so both beams have to support equal weight, and the reduction of safety will be equal for both, by a factor of $\frac{12}{13} \approx 0.92$. The cable and strut solution is the optimum. The stress volume is Pa; the Maxwell number is zero; the volume will be $Pa\left(\dfrac{1}{f_+} + \dfrac{1}{f_-}\right) = \dfrac{18}{17f_-}Pa$; and the weight will be $Pa\left(\dfrac{\rho_+}{f_+} + \dfrac{\rho_-}{f_-}\right) = \dfrac{27\rho_-}{17f_-}Pa$.

In Hemp or Prager' realms, the optimal solution is the cable alone, with a volume of $\dfrac{1}{17f_-}Pa$ in Hemp's realm, lesser than Maxwell's volume, but now the upper beam will be overloaded with P, so it will lessen its safety by a factor of $\frac{6}{7} = 0.86$. In Prager's realm, the optimal solution has a weight of $\dfrac{10\rho_-}{17f_-}Pa$, lesser than Maxwell's weight, but now the upper beam will be overloaded again with P.

The self-weight of solutions with normal steel and wood will be approximately 4 thousandths of Q. The optimal solution selected will depend on the purpose of the designer.

6 EXAMPLES

In the sequel, via three examples, the relative importance of the outlined differences between competing approaches to the structural design in terms of cost and structural shape will be evaluated.

6.1 Illustrative example of Rozvany in an article of 1996

In the article 'Some shortcomings in Michell's truss theory', Rozvany looks for an explanation of the discrepancy between Michell's and Hemp's criteria Rozvany [37]. For this reason, he reproduces Michell's Eq. (2) exactly, which gives the geometrical volume of a full-stressed truss, speaking of '*any* statically determinate truss' (p. 244).

$$\frac{\mathcal{V}^+}{f_+} + \frac{\mathcal{V}^-}{f_-} = V \tag{4}$$

Nevertheless, it should be stressed again that Michell always works with *given external forces in equilibrium*, without any mention of kinematic conditions nor does he refer to any static condition for the competing frames.

After that, Rozvany [37, p. 244] adds that 'the volume [of the truss] can also be calculated by means of the "dual formula"'.

$$V = \frac{1}{2}\left(\frac{1}{f_+} + \frac{1}{f_-}\right) \cdot \mathcal{V} \tag{5}$$

Furthermore, Rozvany also argues that 'Examples of these volume equations are given by Michell. . . '; which is true except in Michell's example 3 [1, 596–597, Fig. 4] as its corresponding equation does not fulfill the equation (5). On the contrary, said example shows clearly the general equation. This equation, which gives the volume as a function of stress volume and Maxwell number, has been well established by many authors (e.g., Hemp [8, 4, Eq.(7)] Cox [2, 87, Eq.(121)], etc) as that of Michell's Lemma (lemma 8), Eq. (3) herein, see page 5.

Note that if either $\mathcal{M} = 0$ or $f_+ = f_-$ are fulfilled, the Michell equation (3) leads to Rozvany's 'dual formula' Eq. (5), i.e., Rozvany's equation is a particular case of the Michell equation. This is why this equation appears in many of the examples of Michell's original article (in these examples $\mathcal{M} = 0$).

Rozvany [37] attempts to show in section 3 of his own article an 'illustrative example', see Fig. 2 and Table. 2, however it is rather misplaced due to it having displacement constraints (a 'fixed boundary' problem) so Michell's theory is not applicable. Moreover, Rozvany uses (5) with unequal allowable stresses, but by coincidence, the Maxwell number of the conjectured Michell truss is null in the selected example—Rozvany [37, Fig. 1b], from Prager and Rozvany [7, Fig. 1]—, so he has no chance to detect any discrepancy between his primal and incorrect dual formulae. Unfortunately as Rozvany's purpose is 'to provide a constructive explanation of the apparent discrepancy between Hemp's and Michell's criteria'. While with this inadequate equation in mind, Rozvany states that the 'discrepancy' is not 'apparent', because in fact in his view it exists. Therefore the 'critical examination of Michell's proof' in section 4 of Rozvany's article makes no sense as it is based on all these misunderstandings and mislead outcomes of the *unique* example that Rozvany considers.

His conclusion is that 'for unequal permissible stresses Michell's optimality conditions are only valid for a highly restricted class of support conditions' and that Michell 'was not aware of the limits of validity of his theory'. However, when reading Michell's paper and examining his examples carefully, it is clear that Michell understood the profound, general meaning of Maxwell's findings.

It is worth examining in detail the illustrative example mentioned. The original, fixed-boundary problem (Figure 2(a)) consists in how to translate a vertical useful load P to a wall a horizontal distance L apart.

Rozvany [37, p. 245] derives an optimal solution from Michell's optimal condition, see Fig. 2(c), despite that this condition is not applicable here as the problem is not of Maxwell type. (For this solution $\mathcal{M} = 0$.)

Anyway Rozvany found that Hemp's criterion leads to a better solution (Fig. 2(d), with $\mathcal{M} = 2/\sqrt{3}$). Rozvany neither calculates $\mathcal{M}$ —as Authors do—, so he could not notice or realise that the two solutions are not commensurable by $\mathcal{V}$ in the scalar metric proposed by Maxwell and Michell. If Hemp's solution was assumed to be the best one, a question would arise again: what is the cost of the mechanical role of the wall?

In order to answer this question, a simple Maxwell problem could be this one: to transport horizontally the load from the point A to some point of the wall, say to B (Fig. 2(b)). It is suggested that the reader imagines a symmetrical frame into the wall, adds this to the original, and operates with the 2-load resulting frame and dividing any metric result —$\mathcal{M}$, $\mathcal{V}$, volume, etc.— by two at the end.

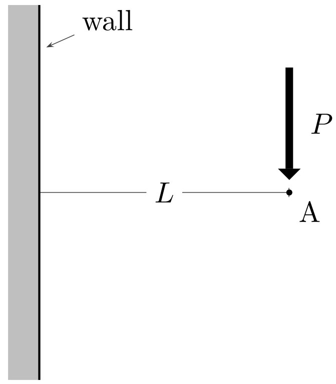
The cost of transporting a part of the load from the upper joint —in the wall— to B is the cost of BU member, and conversely from the bottom, the cost of DB. Making the appropriated corrections it results that $\mathcal{M} = 0$ in both solutions (i.e., they are now commensurable in Maxwell's world), the volume of the solution '1b' (Fig. 2(c)) is proportional to 6, and the volume of '1c' (Fig. 2(d)) is greater, proportional to $13/\sqrt{3} \approx 7.51$ (the optimal value, is proportional to $2 + \pi/2 \approx 3.57$), being both proportional to the respective values of $\mathcal{V}$, since $\mathcal{M} = 0$. See table 2 for a detailed account.

What is the discrepancy? From the Authors point of view, there is none! Hemp's criterion operates on non-Maxwell problems and Michell's criterion on Maxwell ones. In other words: Hemp's criterion is useful for problems in a restricted environment as could be domestic housing, where walls are specified for acoustic or insulation conditions or other non-structural requirements and are supposed to have enough strength for any 'domestic' load. However, Michell condition is relevant for usual engineering problems, where the wall, as a boundary condition, is a convenient trick for the structural analysis, denoting a symmetric axis or so; or generally speaking it will be defined, analysed and checked for structural requirements. Indeed, this process gives as a result the cost of the wall—the BD and BU members cost in our version—, which constitutes a significant part of the whole cost of the structure.

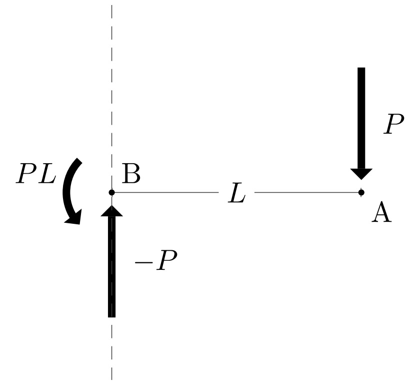
6.2 Parabolic arches

In this example, the problem under consideration is the design of a bridge with a horizontal row of length L, suspended from a parabolic arch with a funicular layout, see Figure 3. The arch is supposed to be compressed without bending action. The only applied load is a uniform one, w, considering negligible self-weight. The curve of the parabolic arch is:

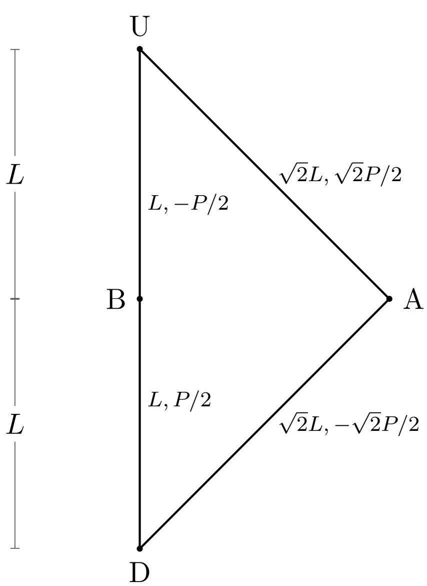
$$y(x,h) = h \times \left\{ 1 - \left(\frac{x}{L/2} \right)^2 \right\}$$



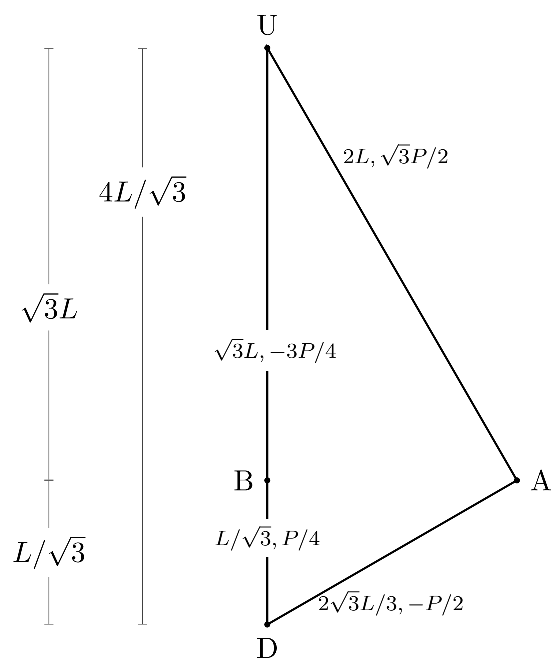
(a) Rozvany's problem



(b) Related Maxwell's problem



(c) Solution '1b'



(d) Solution '1c'

Figure 2. **Illustrative example after Rozvany [37].** Top, the two problems. Bottom, two solutions (original Fig. 1b and Fig. 1c in [37]); for each member the (l,e) couple is given; note that the two figures of the frames are at the same time the vectorial polygons of forces, i.e., if in (d) the scale of the drawing is such that DU$=P$, the internal force of each member can be measured directly in the drawing. P is the 'useful load' and L the problem's size. For Rozvany's problem the design has *only* two members: AU and AD; the simplest Maxwell's Frame has at least four, i.e., BU and BD must be added. The allowable stress is f in tension and $f/3$ in compression, hence $k^+ + k^- = 4/f$ and $k^+ - k^- = -2/f$ must be used in Eq. (3).

Table 2. Data for the problems and designs of Fig. 2

Design:		Fig. 2(c)	Fig. 2(d)
Members's data:	$\mathcal{V}^+(\mathrm{AU})$	PL	$\sqrt{3}PL$
	$\mathcal{V}^+(\mathrm{BD})$	$\frac{1}{2}PL$	$\frac{1}{12}\sqrt{3}PL$
	$\mathcal{V}^-(\mathrm{AD})$	PL	$\frac{1}{3}\sqrt{3}PL$
	$\mathcal{V}^-(\mathrm{BU})$	$\frac{1}{2}PL$	$\frac{3}{4}\sqrt{3}PL$
Rozvany's problem:	$\mathcal{V}$	$2PL$	$\frac{4}{3}\sqrt{3}PL$
	$\mathcal{M}$	0	$\frac{2}{3}\sqrt{3}PL$
	volume	$4\frac{PL}{f}$	$2\sqrt{3}\frac{PL}{f}$
Maxwell's version:	$\mathcal{V}$	$3PL$	$\frac{13}{6}\sqrt{3}PL$
	$\mathcal{M}$	0	0
	volume	$6\frac{PL}{f}$	$\frac{13}{3}\sqrt{3}\frac{PL}{f} \approx 7.51\frac{PL}{f}$
	$\dfrac{\mathcal{V}_{\mathrm{UBD}}}{\mathcal{V}}$	$\dfrac{1}{3}$	$\dfrac{5}{13}$

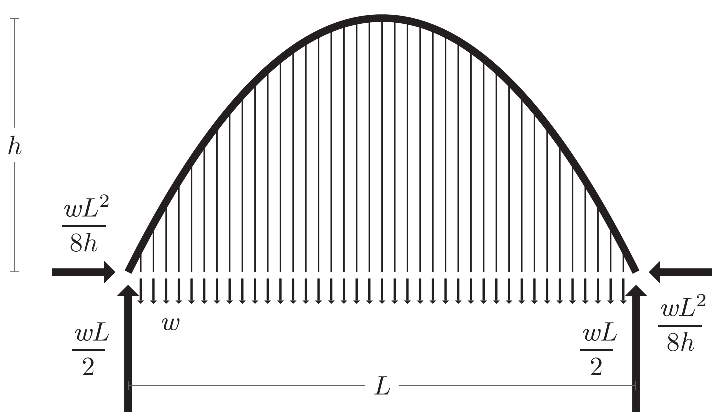


Figure 3. Parabolic arch

Without bending action, the horizontal component of the internal force in the arch is $\dfrac{wL^2}{8h}$ and constant, therefore, the internal force in the arch is:

$$\mathbf{e}_{\mathrm{arch}} = \frac{wL^2}{8h} \times \sqrt{1 + y'(x,h)^2}$$

Table 3. Optima for "classic" arches

$\dfrac{f_+}{f_-}$	$\dfrac{h_{\text{opt}}}{L}$	$\dfrac{V_{\text{opt}}}{wL^2 \div f_-}$	$\dfrac{H_{\text{opt}}}{wL}$
0.20	0.176	1.414	0.707
0.40	0.231	1.080	0.540
0.60	0.265	0.942	0.471
0.80	0.288	0.866	0.433
1.00	0.306	0.816	0.408
2.00	0.353	0.707	0.353
3.00	0.375	0.666	0.333
4.00	0.387	0.645	0.322
5.00	0.395	0.632	0.316

and the internal force in vertical ties will be w, measured by unit of length.

For the sake of brevity, it will be assumed that the ground is as strong as the structural material. Therefore there is no need of foundation in respect to the vertical reaction, only of appropriate connections between the structure and the ground considering the horizontal reaction. Furthermore, it is assumed that the connection is for simple contact, i.e., the friction force between the arch and the ground will equilibrate the horizontal reaction.

6.2.1 A classical approach: optimization of material volume

If the compression material has an allowable stress f_- and the tension material, f_+, the material volume of arch and vertical ties will be at least:

$$V(h) = \frac{1}{f_-}\frac{wL^2}{8h} \times \int_{-L/2}^{L/2} (1 + y'(x,h)^2)\,\mathrm{d}x + \frac{1}{f_+}\int_{-L/2}^{L/2} w \times y(x,h)\,\mathrm{d}x$$

taking into account the arch and the vertical ties. This problem was studied by many authors (see, v.g., [51, 52]).

Considering as a fixed boundary problem, with pinned supports at both edges of the arch, the minimal material volume is obtained from $\partial V(h)/\partial h = 0$:

$$h_{\text{opt}} = \frac{\sqrt{3}}{4}\sqrt{\frac{f_+}{f_+ + f_-}}L$$

In Table 3 the optimal height and other parameters are shown for some values of f_-/f_+. As it is obvious for the expression for h_{opt}, the optimal proportion of the arch, $h \div L$, varies with $f_- \div f_+$, as does the horizontal reaction.

The problem with this approach deals with the cost of the horizontal reactions. Consider a friction coefficient μ between the arch and the support surface. A horizontal reaction up to $\mu wL \div 2$ will be free-of-cost. A normal value for μ is, v.g., 0.3, so a horizontal reaction up to $0.15wL$ does not change the above results. But this figure is surpassed in all cases of Table 3. So the cost of the horizontal reaction matters.

It is worthwhile examining an actual example as the Akashi Kaikyo Bridge [53, 54] . The total load (useful load plus self–weight) is about 2 200 MN with a total length of 3 910 m. The anchorage of main cables is of gravity type: gravity anchorage relies on the mass of the anchorage itself to resist the tension of the main cables, i.e., on friction between foundation and soil. Due to the designed shape, the required anchorage horizontal force at both ends is about 920 MN, and it was obtained with abutments. The anchorage body has about 140 000 m^3 of concrete, i.e., about 3 100 MN of weight, and that means a net friction coefficient of about 0.32. That means that the weight cost of horizontal reactions, 6 200 MN, is 1.59 times the weight of the bridge.

Returning to the parabolic arch, the case $f_- = f_+$ is studied. Assume, for example, $V(0.4L) = 0.85 \times wL^2 \div f_-$ and $H(0.4L) = 0.31 \times wL$. Is this solution worse than the claimed optimum $h = 0.306L$? In fact, if the horizontal reaction cost is null, it is. In any other case, it depends on this cost as the claimed optimum requires a greater reaction, 0.40 versus 0.31, or 0.25 versus 0.16 taking into account the free friction with $\mu = 0.3$.

6.2.2 A Maxwell-Michell approach: optimization of the stress volume

In the sequel, the Maxwell-Michell approach is used again. The geometry and internal force of the arch are the same aforementioned. The stress volume of the arch will be

$$\mathcal{V}^{\mathrm{arch}}(h) = \frac{wL^2}{8h} \times \int_{-L/2}^{L/2} (1 + y'(x,h)^2)\,\mathrm{d}x$$

The reactions at the two ends of the arch are a vertical external force $R = wL/2$ and a horizontal one $\dfrac{wL^2}{8h}$. The suspension, vertical ties to transmit w to the arch needs a stress volume equal to:

$$\mathcal{V}^{\mathrm{v}}(h) = \int_{-L/2}^{L/2} w \times y(x,h)\,\mathrm{d}x$$

The design problem corresponds to an infinite number of Maxwell problems: one for each selected set of reactions. Of course, the vertical reaction is fixed, R, but the designer can select any value for the horizontal reaction on the whole structure, namely H. Considering a set of Maxwell problems depending on a parameter α, such that the horizontal reaction would be prescribed as $H = \alpha \times R$ in each problem belonging to the set.

As the friction is free-of-cost, H will be free-of-cost up to a value $\mu \times R$ where μ is the friction coefficient. The cost of the rest of H above this value depends of the very design of the support. For simplicity, assessing this cost as similar or proportional to the cost of a virtual tie (or a strut)

(1) of length L with internal force of $\operatorname{abs}(H) - \mu \times R$. The stress volume of the reactions can be [so] computed as

$$\mathcal{V}^H(h,\alpha) = \begin{cases} 0 & \text{if } \operatorname{abs}(H) \leq \mu \times R \\ (\operatorname{abs}(H) - \mu \times R) \cdot L & \text{any case else} \end{cases}$$

If the arch and vertical ties were to be a Maxwell structure, the reaction at the two arch ends must have the components R and $\dfrac{wL2}{8h}$, see above. So if $H \neq \dfrac{wL2}{8h}$, the structure needs a real horizontal bar, tie or strut—which is different from the virtual bar used for estimate the reaction cost—, with a stress volume:

$$\mathcal{V}^{\mathrm{h}}(h,\alpha) = \operatorname{abs}\left(H - \frac{wL^2}{8h}\right) \times L$$

Thus, a simple optimization problem stems:

$$\min_{h,\alpha} \mathcal{V}(h,\alpha) \qquad \text{with } \mathcal{V}(h,\alpha) = \mathcal{V}^{\text{arch}}(h) + \mathcal{V}^{\text{v}}(h) + \mathcal{V}^{\text{h}}(h\alpha) + \mathcal{V}^H(h,\alpha)$$

Note that this formulation is general. Although, of course, the definition of the cost of the horizontal reaction H, $\mathcal{V}^H(h,\alpha)$, is tentative: other models will lead to different results. The optima obtained with Cobyla algorithm [55] are shown in Table 4. The important points are these:

1. The optima do not depend on the allowable stresses, not the height h neither the optimal reactions, $\alpha \times wL \div 2$. Of course the material volume will be different for different materials, but not the shape that solely depends on the friction coefficient, μ.
2. For friction lesser than $\mu = 0.5$, the design criterion is simply: take a height of $\sqrt{3}L \div 4$ and design the reaction H according to the reaction that a funicular arch requires, discounting the friction part of the reactions cost.
3. With very high friction, $\mu > 0.9$, as the reaction H is free, take the height that optimizes the arch and vertical ties cost.
4. With medium friction, there is no clear criterion so one has to explore via many Maxwell problems.
5. There are no coincidences between the two approaches (compare Table 3 and 4).

6.3 Hemp's arch for uniform load

Hemp [56] proposes a parametric layout for the problem of minimal volume with uniform load w between two pinned supports, i.e., a fixed boundary problem. Nowadays, this problem is known as the 'optimal girder problem' [42]. In this reference, there is a very comprehensive review of the numerical or analytical solutions obtained till now: numerical solution for the case $f_+ = f_-$ [42, section 4.16.2], analytical solution for the case $f_+ = f_-$ [57], analytical solution for the case $f_+/f_- \leq 0.4177$ [47]. Recently general, near-optimal layouts has been proposed, [see 43, 19]. Currently, there is no known general, analytical, optimal solution for this problem, although.

Table 4. Optima for "total cost" arches, including reactions cost. ($\mathcal{V}^{\text{str}} = \mathcal{V}^{\text{arch}} + \mathcal{V}^{\text{v}} + \mathcal{V}^{\text{h}}$). Compare results for $\mu \geq 0.90$ with those of Table 3 corresponding with $f_+/f_- = 1$.

μ	h_{opt}/L	α_{opt}	$\mathcal{V}_{\text{opt}}/(wL^2)$	$\mathcal{V}^{\text{str}}/(wL^2)$	$\mathcal{V}^{H}/(wL^2)$
0.00	0.433	0.430	1.154	0.939	0.215
0.10	0.433	0.430	1.104	0.939	0.165
0.20	0.433	0.430	1.054	0.939	0.115
0.30	0.433	0.432	1.004	0.938	0.066
0.40	0.433	0.439	0.954	0.935	0.019
0.50	0.433	0.542	0.904	0.883	0.021
0.60	0.412	0.606	0.856	0.852	0.003
0.70	0.290	0.860	0.897	0.817	0.080
0.80	0.287	0.869	0.852	0.818	0.034
0.90	0.285	0.876	0.818	0.818	0.000
1.00	0.285	0.876	0.818	0.818	0.000

The Hemp layout, see Fig. 4, consists of an arch, **OAB**, vertical hangers (as **DE**) between the curve **ODB** and the line **OM**, and a family of orthogonal arches and curved hangers (as **OC** and **AD**) into the region **OABDO**. The main parameter is the slope of the arch at **O**, ϕ_1, that is determined with the condition that a displacement field exists such that the absolute deformation of all the members is equal and its sign equal to the corresponding internal force, i.e., a fully-stressed design. Hemp determines its value when the allowable tensile and compressed stresses are equal. He concludes that his solution, albeit better than the traditional one with parabolic arch and vertical hangers, is suboptimal because it does not fulfil his own optimality criterion [20], since the maximum absolute strain somewhere in the field is greater than this value in the members. Pichugin et al. [47] generalize Hemp's equations for any ratio of the allowable stresses, f_+/f_-, showing that for $f_+/f_- \leq 0.4177$, the resultant layouts from the Hemp family are absolute optima, according to Hemp's criterion.

The method of Pichugin et al. [47] can be summarized as follows: for each value of $f_+/f_- \in [0, 1]$, the value of ϕ_1 is the numerical solution of their Eq. (17). Then, the height h of the solution is determined with their Eq. (6) and finally the volume of material is computed with their Eq. (39). Remember that these values are also optimal if $f_+/f_- \leq 0.4177$, for the 'fixed boundary' problem.

The pinned supports are at the point **O** and at its corresponding symmetrical point with respect to the **MB** line. For each *feasible* solution ϕ_1 there is a vertical reaction $Y = wl$, being l the half-span, and the horizontal reaction X can be determined by simple equilibrium equations of the

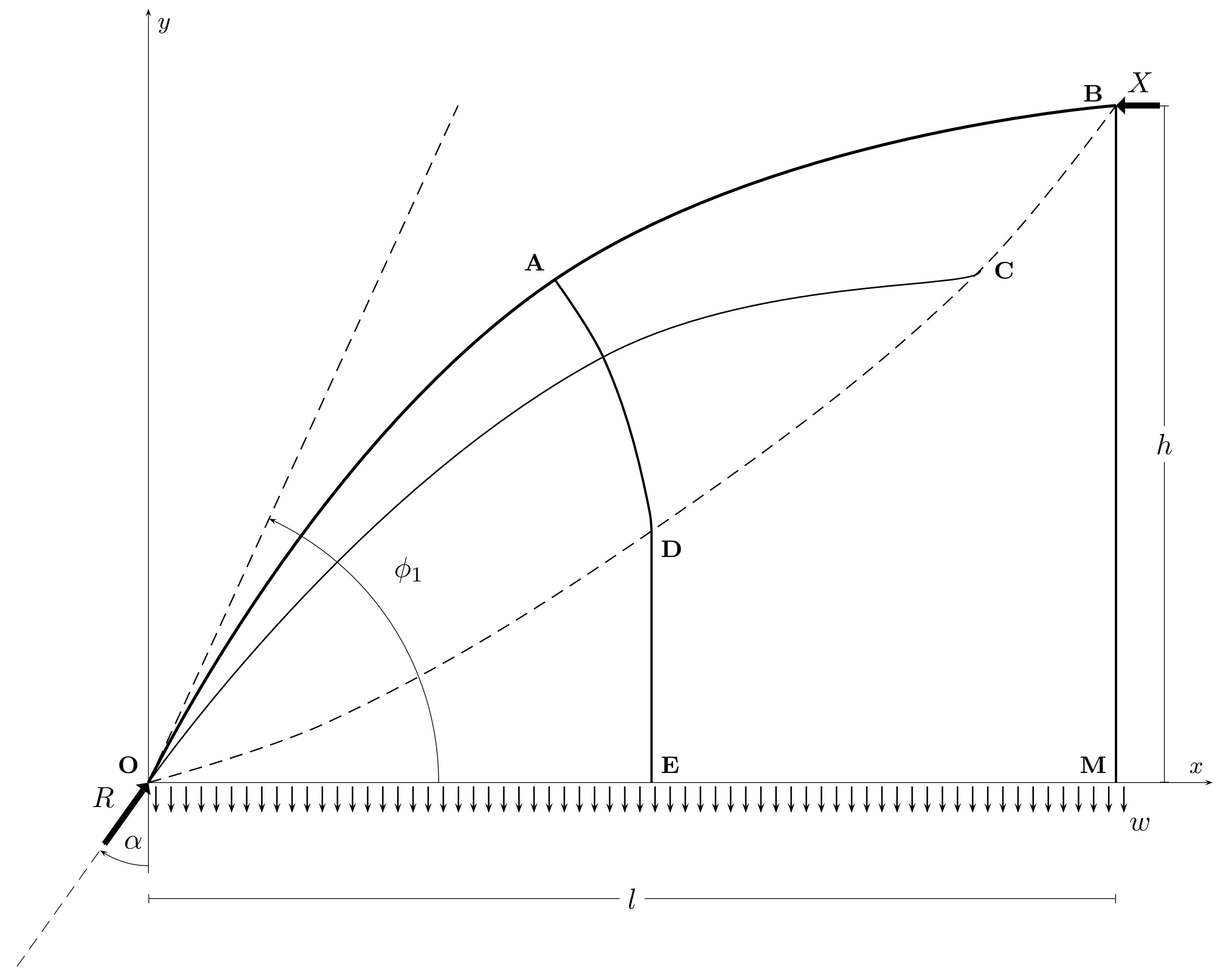


Figure 4. **Half-span of the parametric layout proposed by Hemp [56].** The dotted line ODCB is the boundary between the region occupied by the vertical hangers and the region occupied by the orthogonal net of curved bars.

half-solution:

$$X = \frac{wl^2}{2} \cdot \frac{1}{h(\phi_1)} = Y \cdot \frac{l}{2h(\phi_1)} \tag{6}$$

Therefore, the angle α and magnitude R of the oblique reaction at **O** will be:

$$\alpha(\phi_1) = \arctan \frac{l}{2h(\phi_1)} \qquad R(\phi_1) = \frac{wl}{\sqrt{1+\tan^2\alpha}} \tag{7}$$

As the reaction in the pinned supports depends on ϕ_1, the optimal design problem (coupled with the mathematical problem of Hemp) can be stated as 'finding a solution ϕ_1 with minimal volume, i.e., minimal sum of the foundation volume and the arch volume.' (Outlining that herein

Table 5. Feasible solutions for Hemp's arch problem
Comparison for two solutions with $f_+/f_- = 0.2$ and prismatic foundations orthogonal to reactions (soil cohesion equal $0.002f_-$, internal friction angle of 28^{o}, $w/(f_-l) = 0.0001$).

Solution	ϕ_1 (°)	h/l	$\frac{V_{\text{arch}}}{wl^2/f_-}$	X/wl	R/wl	α (°)	$\frac{V_{\text{abutments}}}{wl^2/f_-}$	$\frac{V_{\text{total}}}{wl^2/f_-}$
PTG	38.872	0.36648	5.5887	1.3643	1.9650	53.760	16.302	21.891
Hemp	63.126	0.67687	6.5117	0.7387	1.5955	36.452	5.7580	12.270

'arch's volume' stands for the whole pinned structure, that is the arch, the fan and hangers families and the bottom tie in the cases of hangers not being vertical.) The functional to be minimized is then:

$$V_{\text{total}} = V_{\text{abutments}}(R(\phi_1), \alpha(\phi_1)) + V_{\text{arch}}(\phi_1) \tag{8}$$

To this end, the shortcoming of the theory of Hemp [20] is simply that the volume of the foundations is not counted in any way, so generally a solution of minimal structure volume, but of sub-optimal overall volume, can be found.

According to this, two special cases must be distinguished: (i) For problems of very small size, the design of abutments will be restricted to minimal dimensions for practical reasons and $V_{\text{abutments}}$ will be constant. (ii) For a Maxwell problem, R and α are constant and so $V_{\text{abutments}}$.

In the case (i) it can be considered a constant abutment so Hemp's theory is applicable—providing that the minimal abutment will have enough support capacity— and the optimal shape will vary with f_+/f_- [2, p. 116]. For this case the solutions of Pichugin et al. [47] has practical interest because some kind of 'existing structure' is used, as Cox named the given abutments.

In the case (ii), it is also possible to consider a constant abutment but Hemp's theory is not applicable, as the condition of constant reaction is incompatible with kinematic support conditions. However, recalling Michell's original theory, the optimal shape will be independent of f_+/f_- , just depending on R and α. In this case the equations of Pichugin et al. [47] to determine the optimal value for ϕ_1 are not appropriated.

For any other case, the two volumes must be taken into account and for the time being there is not a sound optimality-criterion theory for this target. This is the case with normal bridge design, where abutment shape is part of the whole design problem. Anyway, Michell's theory can be useful for each pair (R, α). So the practical design problem corresponds actually to a *set of Maxwell's problems*, each defined by a couple (R, α), i.e., the Michell formulation can help to determine optimal solution for each (R, α), but the designer has to look for the optimal (R, α) comparing the optimal V_{total} of each Maxwell's problem in the set.

Hereafter, only the total volume of two feasible designs is computed, illustrating that the optimal solution for the fixed boundary problem is not the optimum for the design problem.

6.3.1 Volume of the arch in a simple case

Let be f_+/f_- equal to 0.2. If ϕ_1 is the claimed optimal value for the given f_+/f_-, the Eq. (39) from [47] to calculate $V_{\text{arch}}(\phi_1)$ can be used.

Each value of ϕ_1 defines a Maxwell problem (R, α), and the two well-known equations of Corollary 6 and Michell's Lemma (Lemma 8) can be used:

$$\mathcal{V}^+(\phi_1) - \mathcal{V}^-(\phi_1) = \mathcal{M}(\phi_1); \qquad \frac{\mathcal{V}^+(\phi_1)}{f_+} + \frac{\mathcal{V}^-(\phi_1)}{f_-} = V(f_+, f_-, \phi_1) \tag{9}$$

being $\mathcal{M}(\phi_1) = -2 \cdot X(\phi_1) \cdot l$. Then, knowing the volume for any ratio f_+/f_- and corresponding ϕ_1, from (9) the stress volume can be computed:

$$\begin{aligned}\mathcal{V}^+(\phi_1) &= \frac{f_- f_+ V(f_+, f_-, \phi_1) + f_+ \mathcal{M}(\phi_1)}{f_+ + f_-} \\ \mathcal{V}^-(\phi_1) &= \frac{f_- f_+ V(f_+, f_-, \phi_1) - f_- \mathcal{M}(\phi_1)}{f_+ + f_-}\end{aligned} \tag{10}$$

Therefore, the procedure will be as follows: for each ϕ_1 and with the ratio $(f_+/f_-)_{\text{opt}}$ for which ϕ_1 is optimal for the original problem of [47], computing V_{opt} with Eq. (39) of [47], X, R and α with Eqs. (6) and (7) here, and $\mathcal{V}^+$ and $\mathcal{V}^-$ with Eq. (10). Finally, with the ratio $f_+/f_- = 0.2$ computing the corresponding volume V_{arch} with the second of (9). This procedure shows that analytical results from fixed boundary problems can be useful for the corresponding free load version.

6.3.2 A model for the foundation

In order to evaluate V_{total} with (8) it is necessary to establish a model for the foundation. A standard, simple model with the same material as the structure, and with a standard cohesive soil will be used. The abutments will be a simple, prismatic body of base A and depth d; it will have a square contact surface $A = a \times a$ orthogonal to the direction of R, i.e., this surface has a slope $\tan\alpha$ with the horizontal plane. The standard soil has an effective cohesion of $0.002 f_-$ and an effective stress angle of internal friction of 28º.

Then, the allowable stress of the soil under foundation is [58, Art. 4.4.7.1.1.8]:

$$\sigma_S \approx f_-(0.0212\alpha^2 - 0.0798\alpha + 0.0693) = f_- \times F_S(\alpha) \tag{11}$$

being α expressed in radians and accounting only the cohesion term to keep the model simple. The term due to specific weight of soil depends on the value of R and to add it leads to the same conclusion that Authors get later, but increasing the difference of volumes between the solutions analysed. Whereas, the term due to overburden pressure is zero as foundations considered are superficial.

The side of the square base is $a = \sqrt{R/\sigma_S}$, and the depth must be $d \geq \lambda a$, being λ a constant for each shape dependant on the ratios σ_S/f_- —given by (11) for each $\alpha(\phi_1)$— and f_+/f_-:

$$\lambda = \sqrt{\Phi \frac{\sigma_S/f_-}{f_+/f_-}} \quad \text{with} \quad f_+/f_- \leq 1 \tag{12}$$

being Φ a shape factor, equal to 3 for a prismatic foundation. Notice that although the prismatic body is a sub-optimal shape, using an optimal shape will only change Φ and being this factor independent of R or α, this change would decrease the cost of foundations by the same factor, $\sqrt{\Phi}$, for all feasible solutions. Hence the use of the prismatic body has no influence on the already presented argument and keeps the example simple.

Additionally, expressing $V_{\text{abutments}}$ in terms of wl^2/f_-:

$$V_{\text{abutments}} = \frac{wl^2}{f_-} \cdot \left\{ \frac{2\sqrt{3}}{F_S(\alpha)} \left(\frac{R}{wl}\right)^{\frac{3}{2}} \sqrt{\frac{w}{f_+ l}} \right\} \tag{13}$$

As the volume of foundations depends on the ratio $w/(f_+ l) = w/(f_- l)/(f_+/f_-)$ the ratio $w/(f_- l)$ must be fixed to be able to compare different solutions.

6.3.3 Comparison of two designs

According to [47] the method, labelled 'PTG' in table 5, provides the 'optimal' solution with respect to the volume of the arch. This 'optimal' solution is completed by the Authors with the volume of the foundations calculated with (13), with a value of the ratio $w/(f_- l)$ equal to 0.0001, corresponding with $w = 100$kN/m, $l = 100$m and $f_- = 8\,800$kPa. The volume of the foundations is almost three times the volume of the arch, namely, the cost of the foundations cannot be ignored.

Volumes, according with Hemp's original layout, have also been computed. Results, as expected, show that the volume of the arch is greater than the one of PTG solution, but being the reaction R and the angle α lesser, the foundation volume results lesser too. As a result, the total volume of the 'Hemp' design is 56.1% of the PTG one: the former is better than the latter. It is clear that the latter is not an optimum, although obviously the former is not either.

It should be noted that although our example is simple for the sake of brevity, more complex models of the soil strength or an alternative model of foundations will lead to similar conclusions. In the most recent research into the subject of optimal bridge forms, it can be read that 'when modelling a bridge containing end spans, a cost could potentially be ascribed to the resultant horizontal reaction forces, which would in practice need to be carried by costly anchorages' [40], and 'it can be observed that the horizontal reaction force generated at each end of the bridge is very substantial, having a magnitude of over half that of the imposed deck-level load. Resisting such a force in practice would likely be very costly' [41].

7 CONCLUSION

A sound structural design theory has to account for the whole cost of materials that are necessary for a given target, frequently defined by given useful loads, a structure size and a set of surfaces where the structure can sit, see [8, 1–2]; that being especially important for an *optimal structural design theory*, of course. This rule has its origin in thermodynamic accounting rules [5] and is suggested by Maxwell [6]. This comprehensive accounting must be the common basis for any approach to structural design theory, including structural optimization, whichever is the function (or functions) to be optimized or the optimization method applied.

The historical review documented in this paper has shown that, Michell [1] following Maxwell, presents a method fully respectful to this rule that can help to find an optimal structure when a set of external forces in equilibrium is given. Although Michell's work has been ignored for decades, Cox [2] recalls it and is able to formulate a sound structural design theory with some extensions, in particular the distinction between the 'free loading' and 'fixed boundary' approaches, pointing out the context where each of them can be useful.

After Cox's work, some other methods for solving mathematical optimization problems derived from structural analysis problems are established—see the classic books of Hemp [20] or Rozvany [59]. However, many of them cannot easily tackle the problem of considering the whole cost of alternative structures and materials involved, despite presenting an interesting advance from a mathematical standpoint.

For that reason, the examples presented herein attempt to show that some mathematical solutions that claimed to be optimal for partial accounting of structural cost, can be useless from a practical perspective, and suggest the following conclusions:

1. The Maxwell-Michell theory allows to take into account the complete cost of the structure from the very beginning of its design process, when the structural problem is adequately posed.
2. There is no actual discrepancy between the theory of Michell and those of Hemp or Prager, normally the latter being only applicable to substructures, as they only take into account a partial cost, not the whole cost of the complete structural solution to the posed problem.
3. The aphorism attributed to Voltaire, *'The best is the enemy of good'* has to be considered seriously in order to incorporate these optimization techniques and findings to every-day work. Nevertheless, many findings of optimization that search into the mentioned common basis can be translated into *design rules* for structural types, and these rules can be perhaps more easily incorporated in every-day work than the optimization methods from which these findings are obtained. Of course, these rules do not guarantee the optimum, however they help very much in obtaining *good* designs.

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CONFLICT OF INTEREST

No potential conflict of interest was reported by the Authors. All the Authors receive a salary from the Universidad Politécnica de Madrid.

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